THE GROUPS DETERMINED BY THE RELATIONS
$S^l = T^m = (S^{-1}T^{-1}ST)^p = 1$

PART I

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In working out the commutator subgroups of the finite groups generated by reflections, I came across a group of order 288 having the abstract definition

$$S^3 = T^3 = (S^{-1}T^{-1}ST)^2 = 1.$$

When I sent this result to Dr. Sinkov, he replied that he was making a special study of such groups. So we agreed to write consecutive papers, his abstract treatment to follow my geometrical treatment.

Groups of the form $S^l = T^m = (ST)^n = 1$, considered for the sake of analogy

A triangle of angles π/l, π/m, π/n can be drawn on a sphere, or in the euclidean plane, or in the hyperbolic plane, according as the number $1/l + 1/m + 1/n$ is greater than, equal to, or less than unity. By reflecting this triangle in its sides repeatedly, we fill the whole sphere or plane with such triangles, which may be shaded or left white, according to their orientation. Dyck[1] showed that the white (or shaded) triangles correspond to the operators of the abstract group

$$S^l = T^m = (ST)^n = 1.$$

It follows that this group is finite when

$$1/l + 1/m + 1/n > 1,$$

and infinite otherwise. More precisely, its order is

$$\frac{2}{1/l + 1/m + 1/n - 1}$$

whenever this number is positive, and is infinite otherwise. Miller[2] proved that each infinite group has an infinite number of finite factor groups.

Very little is known about the infinite groups, save in the euclidean case

$$1/l + 1/m + 1/n = 1.$$

This case is manageable on account of the presence of self-conjugate subgroups generated by translations, whose quotient groups are obtained by identifying

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[1] W. Dyck, *Gruppentheoretische Studien*, Math. Ann., vol. 20 (1882), pp. 1–44.

[2] G. A. Miller, *Groups defined by the orders of two generators and the order of their product*, Amer. Jour. of Math., vol. 24 (1902), pp. 96–100.