COLLECTIONS FILLING A PLANE

By J. H. Roberts

Introduction. In 1928 the author showed[1] that there exists an upper semi-continuous collection G filling a plane S such that every element of G is a bounded continuum not separating S. Later he stated[2] that the elements of G could all be taken to be bounded continuous curves. If M is a bounded continuous curve lying in a plane S and not separating S, either M is an arc or M contains a triod.[3] But any collection of mutually exclusive triods lying in a plane[3] is necessarily countable. Consequently, all of the elements of the collection G of continuous curves filling S, except possibly a countable number, are arcs. In view of this result it seemed likely that there existed an upper semi-continuous collection G filling S such that *every* element of G was an arc. In fact, the author has since stated[4] erroneously that such is the case. The principal object of the present paper is to prove that *there does not exist an upper semi-continuous collection G of arcs filling a plane S.* In view of this result, the fact that there is a collection G, every element of which is a bounded continuous curve not separating S, becomes of more interest, and accordingly an example of such a collection G is given.

Definition. A collection G of closed point sets lying in a metric space is said to be *upper semi-continuous*[5] if for each element g of G and each positive e there exists a positive d such that if x is an element of G and $l(x, g) < d$, then $u(x, g) < e$.

Definition. The element g of G is a *limit element* of a subcollection K of G if for every positive e there is an element x of K distinct from g such that $u(x, g) < e$.

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[1] Fundamenta Mathematicae, vol. 14 (1929), pp. 96–102.

[2] This result was presented to the North Carolina Academy of Sciences, May, 1934, but no published statement of it has appeared.

[3] A triod is the sum of three arcs AP_1, AP_2 and AP_3, each pair having only A in common. Cf. R. L. Moore, *Foundations of Point Set Theory*, Theorem 71, p. 250, and Theorem 75, p. 254. Theorem 75 is stated for a closed and compact set, but the present result obviously follows, since the plane is the sum of a countable number of such sets.

[4] See abstract #196, Bull. Amer. Math. Soc., vol. 41 (1935), p. 330.

[5] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., vol. 27 (1925), pp. 416–428. If M is a point set and P is a point, then by $l(P, M)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are point sets, then by $l(M, N)$ is meant the lower bound of the values $l(P, N)$ for all points P of M, while by $u(M, N)$ is meant the upper bound of these values for all points P of M. It is to be observed that $u(M, N)$ may be different from $u(N, M)$, while $l(M, N) = l(N, M)$. The quantities $l(M, N)$ and $u(M, N)$ are called the lower, respectively upper, distances of M from N.