ON VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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We propose to prove an existence theorem for integral equations of the type

$$f(x) = g(x) + \lambda \int_0^x f(y) d_y K(x, y) . \tag{1}$$

The integral used is the Young-Stieltjes integral over the open interval $(0, x)$. The theorem is no longer true if an integral over the closed interval is used. We assume that $g(x)$ is bounded and Borel measurable on $(0, 1)$ and $K(x, y)$ is subject to the conditions

(A) $K(x, y)$ is Borel measurable as a function of x.

(B) There exists a monotone increasing and bounded function $V(y)$ such that

$$| K(x, y_1) - K(x, y_2) | \leqq | V(y_1) - V(y_2) | .$$

(We shall assume that $V(0) = 0$).

(C)[1] $K(x, y) = K(x, y - 0)$.

(D)[2] $K(x, 0) = 0$.

A function $f(x)$ is said to be a solution of (1) if it is bounded and Borel measurable and satisfies (1).

The essential difference between this problem and that of ordinary Volterra integral equations is that integration by parts of Stieltjes integrals is not permissible, so that it is necessary to obtain a formula which will replace integration by parts.[3] We do this as a lemma.

Lemma. *If $f(x)$ is positive, bounded and Borel measurable and $g_1(x)$, $g_2(x)$ are monotone increasing, bounded and continuous on the left, then*

$$\int_0^1 f(x) d[g_1(x) g_2(x)] \geqq \int_0^1 f(x) g_1(x) dg_2(x) + \int_0^1 f(x) g_2(x) dg_1(x) . \tag{2}$$

Proof. We shall first prove the lemma when $g_1(x)$ and $g_2(x)$ are both step functions. We choose the set of points $\{x_n\}$ composed of all points of discontinuity of $g_1(x)$ and $g_2(x)$. Since the functions are continuous on the left, we have

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[1] The condition (C) entails no loss of generality, for the value of a function $h(x)$ of bounded variation may be changed at every point of discontinuity without changing $\int_0^1 f(x) dh(x)$.

[2] The condition (D) clearly involves no loss of generality.

[3] When $K(x, y)$ is continuous in y, integration by parts is permissible. This case of the problem has been solved by Tamarkin. See abstract, Bull. Amer. Math. Soc., vol. 35 (1929), p. 165.