ON SUBHARMONIC FUNCTIONS

By E. F. Beckenbach

1. **Introduction.** The following theorem, first given by Montel, was completed by Radó.[1]

A necessary and sufficient condition that the non-negative continuous function $p(u, v)$ be of class PL[2] is that for all real constants α, β the function

$$e^{\alpha u+\beta v}\, p(u, v)$$

be subharmonic.

The above theorem has been generalized by Kierst and Saks.[3]

It is the purpose of the present paper to present an immediately equivalent form (§2) of the Montel-Radó theorem, and to give two simple geometric consequences (§4 and §5). Without recourse to the Montel-Radó theorem, the latter of these consequent results has been given previously;[4] it is repeated briefly here because the present setting seems to be its proper one.

2. Lemma. *A necessary and sufficient condition that the non-negative continuous function $p(u, v)$, for (u, v) in some domain D, be of class PL is that for all analytic functions $f(u + iv)$, for (u, v) in D, the function*

$$p(u, v)\, |\, f(u + iv)\, | \tag{1}$$

be subharmonic.

Necessity. If $p(u, v)$ is of class PL, since the absolute value of an analytic function is of class PL, and since the product of two functions of class PL is a function of class PL, it follows that (1) is of class PL, and therefore, à fortiori, is subharmonic.

Sufficiency. If (1) is subharmonic for all $f(u + iv)$, it is subharmonic in

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[1] P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, Journal de Mathématiques, (9), vol. 7 (1928), pp. 29–60, especially, p. 40. T. Radó, *Remarque sur les fonctions subharmoniques*, Comptes Rendus, vol. 186 (1928), pp. 346–348. In proving the sufficiency, Montel assumed continuous partial derivatives of the first and second order; Radó removed this restriction.

[2] A function $p(u, v)$, defined in a domain D, is said to be of class PL in D provided $p(u, v)$ is continuous and $\geqq 0$ in D and $\log p(u, v)$ is subharmonic in the part of D where $p(u, v) > 0$. See E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 648–661 for the definition and elementary properties of these functions.

[3] S. Saks, *On subharmonic functions*, Acta Szeged, vol. 5 (1930–32), pp. 187–193.

[4] E. F. Beckenbach, *A characteristic property of surfaces of negative curvature*, Bull. Amer. Math. Soc., vol. 40 (1934), pp. 761–768.