

NECESSARY AND SUFFICIENT CONDITIONS IN THE MOMENT PROBLEM FOR A FINITE INTERVAL

BY R. P. BOAS, JR.

1. **Introduction.** The moment problem of F. Hausdorff is the determination of necessary and sufficient conditions that a sequence of numbers $\{\mu_n\}$ have the form

$$(1) \quad \int_0^1 t^n d\alpha(t) = \mu_n \quad (n = 0, 1, 2, \dots),$$

where $\alpha(t)$ is required to belong to some particular class of functions. If $\alpha(t)$ is an integral, (1) becomes

$$(2) \quad \int_0^1 t^n \varphi(t) dt = \mu_n \quad (n = 0, 1, 2, \dots).$$

Since for any $\varphi(t)$ which is integrable (in the sense of Lebesgue) the numbers μ_n , if of the form (2), must have the property that $\mu_\infty = \lim_{n \rightarrow \infty} \mu_n = 0$, we shall consider also conditions that a sequence $\{\mu_n\}$ have the form

$$(3) \quad \int_0^1 t^n \varphi(t) dt = \mu_n - \mu_\infty \quad (n = 0, 1, 2, \dots).$$

Stating the problem in the form (3) merely serves to simplify the formulation of some of our results.

Hausdorff¹ has obtained necessary and sufficient conditions for the existence of solutions of (1) and (2) under a variety of conditions on $\alpha(t)$ and $\varphi(t)$. Hildebrandt² has obtained one of Hausdorff's conditions for the moment problem (1) by utilizing the theory of linear operations,^{2a} the polynomials of S. Bernstein, and a classical theorem of F. Riesz on the general form of a linear functional in the space of continuous functions. Professor Widder suggested to me that it should be possible to obtain conditions analogous to Hausdorff's by the method of Hildebrandt, but using, instead of the Bernstein polynomials, two inversion operators which he has developed³ for moment sequences known to have the

Received April 2, 1935. The author is indebted to Professor D. V. Widder for suggesting the subject of this paper, and for valuable advice and encouragement during its preparation.

¹ F. Hausdorff, *Momentprobleme für ein endliches Intervall*, *Mathematische Zeitschrift*, vol. 16 (1923), pp. 220-248.

² T. H. Hildebrandt, *On the moment problem for a finite interval*, *Bulletin of the American Mathematical Society*, vol. 38 (1932), pp. 269-270.

^{2a} By a linear operation we shall understand an additive and continuous operation.

³ D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, *Transactions of the American Mathematical Society*, vol. 36 (1934), pp. 107-200.