ON THE FUNDAMENTAL NUMBER OF A RATIONAL GENERALIZED QUATERNION ALGEBRA

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1. **Introduction.** Let $\mathfrak{A}$ be a rational generalized quaternion algebra, hereafter referred to merely as an algebra. $\mathfrak{A}$ has a basis $1, i, j, ij$:

$$i^2 = -\alpha, \qquad j^2 = -\beta, \qquad ij = -ji,$$

where α, β are integers, neither divisible by the square of a prime. Such a basis will be said to be a normal basis associated with α and β.

Brandt defined the fundamental number d of $\mathfrak{A}$, employing an arbitrarily chosen maximal realm of integrity $\mathfrak{G}$ in his definition, and showed that d is independent of the particular $\mathfrak{G}$ in $\mathfrak{A}$ which is employed, and that two algebras with the same d are equivalent.[1] We shall determine d explicitly in terms of α and β. This gives a simple criterion for the equivalence of two algebras.

Starting with a normal basis, as above, Albert[2] showed by a series of transformations that $\mathfrak{A}$ has such a basis associated with certain integers τ and σ which have the following properties:

(a) τ is a positive prime, $\tau \equiv 3 \pmod 4$;

(b) σ is an integer prime to τ, containing no square factor > 1, and $-\sigma$ is a quadratic residue of τ;

(c) $-\tau$ is a quadratic non-residue of every odd prime factor of σ;

(d) if σ is even, $\tau \equiv 3 \pmod 8$.

From the method by which such a basis is obtained, there is no obvious relation between the initial α, β and the final τ, σ. τ is any one of the infinitude of primes represented by a certain quadratic form, with a finite number of exceptions. σ was not shown to be unique, but if $\mathfrak{A}$ is not a division algebra, it was shown that $\sigma = -1$.

We shall show that $\sigma = d$, and hence is uniquely determined by $\mathfrak{A}$. Also, that τ may be an arbitrarily chosen prime satisfying the four conditions above. We may take τ as the least such prime and thus have a normal basis associated with a pair of integers which are uniquely determined by $\mathfrak{A}$.

2. **The determination of** d. According to Dickson's[3] definition, a set of integral elements in $\mathfrak{A}$ is a set having certain properties R, C, U, M. It may be

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[1] *Idealtheorie in Quaternionentheorie*, Mathematische Annalen, vol. 99 (1928), pp. 9, 12.

[2] *Integral domains in rational generalized quaternion algebras*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 164–76. In particular, see Theorems 2, 3. In this paper, we replace Albert's τ, σ by $-\tau$, $-\sigma$, respectively.

[3] *Algebras and their Arithmetics*, pp. 141–2.