GENERALIZED MINIMAX PRINCIPLE IN THE CALCULUS OF VARIATIONS

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Introduction

In the study of the critical points of a function $f(x_1, \cdots, x_n)$ one naturally begins with the maximum and minimum points. Similarly, the study of the critical extremals of an integral

$$J = \int_{t_1}^{t_2} f(x, \dot{x})\, dt$$

joining two fixed points begins with the properties of minimizing or maximizing extremals and the question of their existence. It was not until recently that a systematic study was made of critical points of functions and critical extremals of integrals which are not necessarily of the minimizing or maximizing type. This study seems to have had its beginning in 1917 in a paper by Birkhoff[1] in which he enunciated his minimax principle. Birkhoff treats only the critical points and critical extremals of the so-called type one. Beginning with a paper in 1925 Morse[2] has developed systematically by the use of Analysis Situs the existence and the relation between critical points and critical extremals of all types. A. B. Brown and a number of others have also written on this subject.[2]

The principal method used heretofore in obtaining the critical point relations is the following. A value b will be called a critical value of our functional $f(P)$ if there is a critical point Q of $f(P)$ such that $f(Q) = b$. We now consider the connectivities of the domains $f(P) \leqq b$, as the constant b varies from the absolute minimum b_0 of $f(P)$ on the domain under consideration. It is found that the connectivities of $f \leqq b$ change only when the variable b passes through a critical value of $f(P)$. The change in connectivity depends upon the type of the critical point P having this critical value. By studying these changes of connectivity one is able to classify the critical points of $f(P)$ and to obtain the critical point relations. This method was used by Birkhoff in order to obtain his minimax principle and by Morse to obtain a complete set of critical point

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[1] *Dynamical systems with two degrees of freedom*, Transactions of the American Mathematical Society, vol. 18 (1917), pp. 199–300.

[2] For references to literature on this subject see Morse, *Calculus of Variations in the Large*, Colloquium Lectures, American Mathematical Society, vol. 18 (1934). Unless otherwise expressly stated, all references to Morse are to his book. See also Morse and Van Schaack, *Abstract critical sets*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 258–62.