# ON CERTAIN FUNCTIONS CONNECTED WITH POLYNOMIALS IN A GALOIS FIELD

BY LEONARD CARLITZ

1. **Introduction.** Let $GF(p^n)$ denote a fixed Galois field[1] of order $p^n$; let $E = E(x)$ denote a polynomial in an indeterminate $x$ with coefficients in $GF(p^n)$. Consider the product $\psi_k(t) = \Pi(t - E)$, extended over all $E$ of degree $< k$, where $k$ is an arbitrary positive integer. We show, to begin with, that the product has the expansion

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} t^{p^{nj}}, \tag{1.01}$$

the coefficients (defined explicitly in §2) having certain properties analogous to those of the binomial coefficients. Of the properties of $\psi_k(t)$, it is evident from the form of (1.01) that, for $c$ in $GF(p^n)$,

$$\psi_k(ct) = c\psi_k(t), \qquad \psi_k(t + u) = \psi_k(t) + \psi_k(u);$$

we accordingly call $\psi_k(t)$ a *linear* polynomial.[2] As a second characteristic property we mention

$$\psi_k(xt) - x\psi_k(t) = (x^{p^{nk}} - x)\psi_{k-1}^{p^n}(t).$$

This relation suggests the study of the operator $\Delta$ defined by

$$\Delta f(t) = f(xt) - xf(t), \tag{1.02}$$

where $f(t)$ is a linear polynomial. See §3.

We suppose next that $k$ in (1.01) becomes infinite; the product $\Pi(t - E)$ must be modified somewhat. Actually we consider

$$\Pi' \left(1 - \frac{t^{p^n-1}}{E^{p^n-1}}\right), \tag{1.03}$$

the product extending over all *primary* $E$, that is, over all polynomials in which the coefficient of the highest power of $x$ is the 1 element of the Galois field. As we shall see, the question of convergence causes little difficulty; we find that the infinite product (1.03) has the expansion

$$\frac{1}{\xi}\psi(t\xi) = \frac{1}{\xi}\sum_{k=0}^{\infty} \frac{(-1)^k}{F_k} t^{p^{nk}} \xi^{p^{nk}}, \tag{1.04}$$

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[1] For the properties of Galois fields assumed here, see L. E. Dickson, *Linear Groups*, 1901, pp. 3–54.

[2] Called $p$-polynomials by O. Ore, Transactions of the American Mathematical Society, vol. 35 (1933), pp. 559–584.