GENERALIZED PERFECT SETS

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1. In an earlier paper[1] the author has defined generalized derived aggregates or K-derivatives for subsets A of a metric space C with respect to an arbitrary class K of closed sets in C. Under this definition, by the K-derivative $K(A)$ of A is meant the set of all points x such that every neighborhood of x contains a subset of A which is not contained in any K-set. The operation of taking K-derivatives may be iterated, giving successive K-derivatives denoted as follows: $A = A_K^0$, $K(A) = A_K^1, \cdots, K(A_K^{n-1}) = A_K^n, \cdots$, and in general $A_K^\alpha = K(A_K^{\alpha-1})$ or $= \prod_{\beta<\alpha} K(A_K^\beta)$ according as α is an isolated or a limit ordinal.

This suggests the following extension of the notion of a perfect set, to which the present paper is devoted. A set of points A will be said to be *K-perfect* provided it is equal to its own K-derivative, i.e., $K(A) = A$. It results at once that, for all classes K, every K-perfect set is closed. In case K is the class of all single points, then the K-perfect sets reduce to the ordinary perfect sets.

Other examples are: (1) the Sierpinski triangle curve[2] is perfect with respect to the class of all simple closed curves, the class of all simple arcs, or the class of all dendrites; but it is not perfect relative to the class of all regular curves; (2) the set E consisting of the curve $y = \sin 1/x$ together with the interval $I = (-1, 1)$ of the y-axis is perfect with respect to the class of all simple closed curves in E, (a vacuous class); but relative to the class of simple arcs in E, this set is not perfect, since its first arc-derivative reduces to I and its second arc-derivative vanishes; (3) in the set F consisting of the curve $y^2 = x^2 \sin^2 1/x$ together with the origin Q, the point Q itself is perfect with respect to the class K of all simple closed curves contained in F; in fact, in this case $F_K^1 = Q$, $F_K^2 = K(Q) = Q$, and so on.

In what follows we shall suppose our space to be separable and metric, and K will denote an arbitrary class of closed sets in this space.

2. Theorem. *Any closed set A is the sum of a K-perfect set and a countable number of sets each of which is the intersection of A with a K-set.*

Proof. Let B be the first ordinal such that $A_K^\beta = A_K^{\beta+1}$. Since the space is separable, β is of the first or second class. Then A_K^β is K-perfect and we have

$$\text{(i)} \qquad A = A_K^\beta + \sum_{0 \leqq \alpha < \beta} [A_K^\alpha - A_K^{\alpha+1}].$$

Now for any α, each point x of $A_K^\alpha - A_K^{\alpha+1}$ is contained in some neighborhood Q^α such that $A_K^\alpha \cdot Q^\alpha$ is contained in some K-set. Whence, by the Lindelöf

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[1] See American Journal of Mathematics, vol. 54 (1932), pp. 169–175.

[2] See Comptes Rendus, vol. 162, p. 629.