CHAIN-DEFORMATIONS IN TOPOLOGY

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In topology one has repeated occasion to consider homotopic deformations of chains. They give rise to a basic boundary relation[1] between the extreme positions of a chain c_p in the homotopy and what might be termed the loci of c_p and of its boundary $F(c_p)$. All the consequences of the homotopy that concern algebraic topology (i.e., boundary relations and the associated homologies) may be derived from the fundamental relation. It seems natural therefore to call *chain-deformation* any scheme wherein two p-chains c_p, c_p' and two other chains that are to take the part of the loci mentioned above, satisfy a boundary relation formally identical with the fundamental relation of homotopy.[2] This notion has already been exploited in a recent paper.[3] We return to it here, first to develop it more fully and then to apply it to the study of the sets that are obtained whenever, in the definition of locally connected sets, singular cells and spheres are replaced by chains. These new sets may be described as locally connected in the sense of homology, and their types correspond substantially to the locally connected types that we have recently investigated.[4] The passage from the first class to the second corresponds also to a substitution of chain-deformation for homotopy.

One of the important results of L2 was the identification of certain locally

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[1] Given for the first time in our Colloquium Lectures, *Topology*, New York, 1930, p. 78.

[2] While chain-deformations have most of the properties that their name suggests, they are essentially different from homotopy. This is clearly seen by noting the different effect in the very simple case of the circuits on an orientable surface of genus $p \geqq 2$. Homotopy leads, in this case, to the non-commutative Poincaré group, chain-deformation to the much simpler abelian group with $2p$ free generators.

[3] S. Lefschetz, *On generalized manifolds* (= L1 in the sequel), American Journal of Mathematics, vol. 55 (1933), pp. 475–499.

[4] S. Lefschetz, *On locally connected and related sets* (= L2 in the sequel), Annals of Mathematics, vol. 35 (1934), pp. 118–139. We call attention to the following errata: p. 119, line 23, replace LC by LC^∞; p. 126, suppress line 3 from bottom; in line 4 from bottom, suppress "convex"; in line 5 from bottom, replace "convex sets of $\mathfrak{H}$" by "spheres"; p.127, line 13, replace K^* by $\mathfrak{K}^*$.

Local connectedness in the sense of homology was introduced by P. S. Alexandroff in his paper: *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Mathematics, vol. 30 (1929), pp. 101–187. See also his recent paper: *On local properties of closed sets*, Annals of Mathematics, vol. 36 (1935), pp. 1–35, §3. The same property for euclidean domains plays a central part in R. L. Wilder's recent work. See in particular his last paper: *Generalized closed manifolds in n-space*, Annals of Mathematics, vol. 35 (1934), pp. 876–903.