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# A NEW ELLIPSOID ASSOCIATED WITH CONVEX BODIES

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Corresponding to each origin-symmetric convex (or more general) subset of Euclidean $n$-space $\mathbb{R}^n$, there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about *every* 1-dimensional subspace of $\mathbb{R}^n$. This ellipsoid is called the Legendre ellipsoid of the convex set. The Legendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics. See Milman and Pajor [MPa1], [MPa2], Lindenstrauss and Milman [LiM], and Leichtweiß [Le] for some historical references.

It has slowly come to be recognized that alongside the Brunn-Minkowski theory there is a dual theory. The nature of the duality between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory is subtle and not yet understood. It is easily seen that the Legendre (and Binet) ellipsoid is an object of this dual Brunn-Minkowski theory. This observation leads immediately to the natural question regarding the possible existence of a dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory. It is the aim of this paper to demonstrate the existence of precisely this dual object. In retrospect, one may well wonder why the new ellipsoid presented in this note was not discovered long ago. The simple answer is that the definition of the new ellipsoid becomes obvious only with the notion of $L_2$-curvature in hand. However, the Brunn-Minkowski theory was only recently extended to incorporate the new notion of $L_p$-curvature (see [L2], [L3]).

A positive-definite $n \times n$ real symmetric matrix $A$ generates an ellipsoid $\epsilon(A)$, in $\mathbb{R}^n$, defined by

$$\epsilon(A) = \left\{ x \in \mathbb{R}^n : x \cdot Ax \leq 1 \right\},$$

where $x \cdot Ax$ denotes the standard inner product of $x$ and $Ax$ in $\mathbb{R}^n$.

Associated with a star-shaped (about the origin) set $K \subset \mathbb{R}^n$ is its Legendre ellipsoid $\Gamma_2 K$, which is generated by the matrix $[m_{ij}(K)]^{-1}$, where

$$m_{ij}(K) = \frac{n+2}{V(K)} \int_K \left(e_i \cdot x\right)\left(e_j \cdot x\right) dx,$$

with $e_1, \ldots, e_n$ denoting the standard basis for $\mathbb{R}^n$ and $V(K)$ denoting the $n$-dimensional volume of $K$.

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