Vol. 104, No. 3 DUKE MATHEMATICAL JOURNAL © 2000

DIFFERENTIABILITY PROPERTIES OF ISOTROPIC FUNCTIONS

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1. Introduction. Let Sym denote the linear space of all symmetric second-order tensors on an n-dimensional real vector space Vect with scalar product. (If Vect is identified with R^n, then Sym may be identified with the set of all symmetric n-by-n matrices.) A function $f : \mathrm{Sym} \to \mathrm{R}$ is said to be isotropic if $f(\mathbf{A}) = f(\mathbf{QAQ}^T)$ for all $\mathbf{A} \in \mathrm{Sym}$ and all $\mathbf{Q}$ proper orthogonals. An isotropic function has a representation $f(\mathbf{A}) = \tilde{f}(a)$, where $\tilde{f}$ is a symmetric function on R^n and $a = (a_1, \ldots, a_n)$ are the eigenvalues of $\mathbf{A}$ with appropriate multiplicities. Clearly, $\tilde{f}(a) = f(\mathrm{diag}(a))$ in any orthonormal basis, and thus if f is of class C^r, $r = 0, 1, \ldots, \infty$, then also $\tilde{f}$ is of class C^r. Ball [1] showed that for $r = 0, 1, 2, \infty$, the converse is also true and conjectured that the converse is true for all r. This was subsequently proved by Sylvester [6] using complex techniques and detailed estimates of the derivatives of eigenvalues. Earlier, Chadwick and Ogden [2], [3] gave formulas for $D^r f$, $r = 1, 2, 3$, in terms of $\tilde{f}$ and its derivatives assuming the differentiability (see also [1]). In this note, I derive the result of Sylvester by elementary means and give a recursive formula for $D^r f$ in terms of $\tilde{f}$ for arbitrary r. I also specialize these formulas to derive the forms of $D^r f$, $r = 1, 2, 3$, which are equivalent to those by Chadwick and Ogden.

2. Notation. Throughout, the indices i, j, k range the interval $\{1, \ldots, n\}$, unless stated otherwise. The direct vector notation is used in [4], [5]. In addition to the notation explained in the introduction, we recall that a second-order tensor $\mathbf{A}$ is a linear transformation from Vect into Vect, with the product of two tensors being the composition of the linear transformations. Furthermore, Orth^+ denotes the proper orthogonal group, and Skew denotes the set of all skew tensors. By a basis in Vect, we always mean an orthonormal basis. Let S_n be the set of all real symmetric n-by-n matrices. Let e_i be the canonical basis in R^r. All vector spaces are finite-dimensional and real.

For a vector space X, we denote by $\mathrm{F}^r(\mathrm{X})$ the vector space of all symmetric r-linear forms $F : \mathrm{X} \times \cdots \times \mathrm{X} \to \mathrm{R}$ on X. The direct notation is used to denote the derivatives (differentials) of functions f defined on a vector space X with values in R. Thus for $x \in \mathrm{X}$, the rth derivative $D^r f(x)$ is a symmetric r-form on X; that is,

Received 19 May 1999. Revision received 3 June 1999.
2000 *Mathematics Subject Classification*. Primary 74A20; Secondary 74B99.
Author's work supported by grant number 201/00/1516 of the Grant Agency of the Czech Republic.