Vol. 104, No. 1 DUKE MATHEMATICAL JOURNAL © 2000

THE GAUSS MAP AND A NONCOMPACT RIEMANN-ROCH FORMULA FOR CONSTRUCTIBLE SHEAVES ON SEMIABELIAN VARIETIES

J. FRANECKI AND M. KAPRANOV

1. Introduction and statement of results. Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\mathfrak{F}$ be a constructible sheaf of $\mathbb{C}$-vector spaces on X. As in other situations, we have the Riemann-Roch problem: express $\chi(X,\mathfrak{F}) = \sum_i (-1)^i \dim H^i(X,\mathfrak{F})$ in terms of some intrinsic geometric invariants of X and $\mathfrak{F}$. One such invariant is the characteristic cycle $CC(\mathfrak{F})$, which is a formal $\mathbb{Z}$-linear combination $\sum_\nu n_\nu[\Lambda_\nu]$ of irreducible conic Lagrangian subvarieties Λ_ν in the cotangent bundle T^*X; see [14] for background. When X is compact, the Riemann-Roch problem has a nice solution, namely (see [13]),

$$\chi(X,\mathfrak{F}) = \big(CC(\mathfrak{F}),[X]\big)_{T^*X}, \tag{1.1}$$

where the right-hand side is the intersection index, in T^*X, of $CC(\mathfrak{F})$ and the zero section $X \subset T^*X$. It can be calculated, for instance, by first deforming X to the graph of a C^∞ 1-form so that the intersection becomes transverse, and then counting intersection points (with multiplicities and signs).

Both the definition of $CC(\mathfrak{F})$ and the formula (1.1) (for compact X) extend to the case when $\mathfrak{F}$ is a bounded constructible complex (i.e., a complex of sheaves with constructible cohomology).

When X is not compact, $\chi(X,\mathfrak{F})$ still makes sense, but (1.1) is not applicable. We face, therefore, an interesting *noncompact Riemann-Roch problem* of finding $\chi(X,\mathfrak{F})$ in terms of invariants intrinsic to X (in particular, not involving the choice of compactification).

The purpose of this paper is to exhibit such a "noncompact Riemann-Roch formula" in a particular class of situations, namely, suppose that $X = G$ is an algebraic group with Lie algebra $\mathfrak{g}$. For $\gamma \in \mathfrak{g}^*$ let ω_γ be the corresponding left-invariant 1-form on G, and let $\Omega_\gamma \subset T^*G$ be its graph. The Ω_γ then form a natural family of deformations of X, and we can use them to make sense of the intersection index in (1.1) even when G is not compact. More precisely, if $\Lambda \subset T^*G$ is an irreducible conic Lagrangian subvariety and if $\gamma \in \mathfrak{g}^*$ is *generic*, then $\Lambda \cap \Omega_\gamma$ consists of finitely many transversal intersection points; their number is denoted $\operatorname{gdeg}(\Lambda)$ and called the *Gaussian degree* of Λ. To explain the name, recall that Λ has the form $T^*_Z X$ for an irreducible subvariety

Received 10 December 1999. Revision received 21 December 1999.
2000 *Mathematics Subject Classification*. Primary 32S60; Secondary 20G20.
Kapranov partially supported by National Science Foundation grant number 9623044.