ARRANGEMENT OF HYPERPLANES, II: THE SZENES FORMULA AND EISENSTEIN SERIES

MICHEL BRION AND MICHÈLE VERGNE

To Victor Guillemin, for his 60th birthday

1. Introduction. Consider a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of linear forms in r complex variables, with integral coefficients. The linear forms α_j need not be distinct. For example, $r = 2$ and $\alpha_1 = \alpha_2 = z_1$, $\alpha_3 = \alpha_4 = z_2$, $\alpha_5 = \alpha_6 = z_1 + z_2$. For any such sequence, D. Zagier [5] introduced the series

$$\sum_{n \in \mathbb{Z}^r, \langle \alpha_j, n \rangle \neq 0} \frac{1}{\prod_{j=1}^{k} \langle \alpha_j, n \rangle}.$$

Assuming convergence, its sum is a rational multiple of π^k. For example (see [5]), we have

$$\sum_{n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0} \frac{1}{n_1^2 n_2^2 (n_1 + n_2)^2} = \frac{(2\pi)^6}{30240}.$$

These numbers are natural multidimensional generalizations of the value of the Riemann zeta function at even integers. A. Szenes gave in [3, Theorem 4.4] a residue formula for these numbers, relating them to Bernoulli numbers. The formula of Szenes [3] is the multidimensional analogue of the residue formula

$$\sum_{n \neq 0} \frac{1}{n^{2l}} = (2\pi)^{2l} \frac{B_{2l}}{(2l)!} = (-1)^l (2\pi)^{2l} \operatorname{Res}_{z=0} \left(\frac{1}{z^{2l}(1 - e^z)} \right).$$

A motivation for computing such sums comes from the work of E. Witten [4]. In the special case where α_j are the positive roots of a compact connected Lie group G, each of these roots being repeated with multiplicity $2g - 2$, Witten expressed the symplectic volume of the space of homomorphisms of the fundamental group of a Riemann surface of genus g into G, in terms of these sums. In [2], L. Jeffrey and F. Kirwan proved a special case of the Szenes formula leading to the explicit computation of this symplectic volume, when G is $\mathrm{SU}(n)$.

Our interest in such series comes from a different motivation. Let us consider first the 1-dimensional case. By the Poisson formula, for $\mathrm{Re}(z) > 0$, the convergent series $\sum_{m=1}^{\infty} m e^{-mz}$ is also equal to $\sum_{n \in \mathbb{Z}} 1/(z + 2i\pi n)^2$. Similarly, sums of products

Received 5 March 1999.
2000 *Mathematics Subject Classification*. Primary 52C35; Secondary 11B68, 40H05.