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A CHARACTERIZATION OF RATIONAL SINGULARITIES

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The main purpose of this note is to present a characterization of rational singularities in characteristic 0. The essence of the characterization is that it is enough to require less than the usual definition.

THEOREM 1. *Let $\phi : Y \to X$ be a morphism of varieties over $\mathbb{C}$, and let $\rho : \mathcal{O}_X \to R\phi_*\mathcal{O}_Y$ be the associated natural morphism. Assume that Y has rational singularities and there exists a morphism (in the derived category of $\mathcal{O}_X$-modules) $\rho' : R\phi_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\rho' \circ \rho$ is a quasi-isomorphism of $\mathcal{O}_X$ with itself. Then X has only rational singularities.*

If ρ' exists, it could be considered similar to a trace operator. In fact, for any finite morphism of normal varieties, ρ' exists because of the trace operator.

Note that for the first statement of Theorem 1, ϕ does not need to be birational. In particular, Theorem 1 implies that quotient singularities are rational, including quotients by reductive groups as in [B, Corollaire]. In the latter case, ρ' is given by the Reynolds operator.

A well-known and widely used theorem states that in characteristic 0, canonical singularities are Cohen-Macaulay and therefore rational (see [E] and [KMM]).

The original proofs are based on a very clever use of Grothendieck duality simultaneously for a resolution and its restriction onto the exceptional divisor and on a double loop induction. Kollár gave a simpler proof in [K2, §11] without using derived categories but still relying on a technically hard vanishing theorem. Recently Kollár and Mori found a simple proof allowing nonempty boundaries. They do not use derived categories either, but restrict to the projective case (see [KM, 5.18]). These proofs are ingenious, but one would like to have a simple natural proof (at least in the “classical” case, when the boundary is empty).

As an application of Theorem 1 a simple proof is given here in the “classical” case, but without the projective assumption. This proof seems even simpler than that of Kollár and Mori. Derived categories and Grothendieck duality are used, but in such a simple way that one is tempted to say that this proof is the most natural one. Note also that everything used here was already available when the question was raised for the first time.

A statement similar to Theorem 1 was given in [K2, 3.12]. Some ideas of the

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