ON THE STRUCTURE OF THE SELBERG CLASS, III: SARNAK'S RIGIDITY CONJECTURE

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1. Introduction. In this paper we deal with a problem, raised by Peter Sarnak, about rigidity in the Selberg class $\mathcal{S}$. Roughly speaking, the problem is as follows: Is every continuous one-parameter family $\mathcal{F} = \{F(s;\xi)\}_{\xi\in\mathbb{R}}$ of functions in $\mathcal{S}$ a shifted family, that is,

$$F(s;\xi) = \prod_{j=1}^{k} F_j\big(s + ih_j(\xi)\big)$$

with $F_j \in \mathcal{S}$ and $h_j(\xi)$ continuous? We need some preliminaries in order to give a precise formulation of the problem and our results. We refer to the survey paper [4] for the basic notation, definitions, and results about $\mathcal{S}$.

Given a function $F \in \mathcal{S}$, we denote by $a_n(F)$ its nth coefficient and write

$$\theta_F = \operatorname{Im} 2\sum_{j=1}^{r}\left(\mu_j - \frac{1}{2}\right),$$

where the μ_j appear in the Γ-factors of a functional equation of $F(s)$. We recall that the shift θ_F is an invariant of $F(s)$ (see [4, Section 8]). Moreover, for every entire $F \in \mathcal{S}$ and every $\theta \in \mathbb{R}$, the shifted function $F_\theta(s) = F(s+i\theta)$ belongs to $\mathcal{S}$. We also recall the Selberg orthonormality conjecture, asserting that if $F, G \in \mathcal{S}$ are primitive functions, then

$$\sum_{p\le x}\frac{a_p(F)\overline{a_p(G)}}{p} = \delta_{F,G}\log\log x + O(1) \text{ as } x \longrightarrow \infty, \quad \text{where } \delta_{F,G} = \begin{cases} 1 & \text{if } F = G,\\ 0 & \text{if } F \ne G.\end{cases}$$

We further recall that under Selberg orthonormality conjecture, $\mathcal{S}$ has unique factorization into primitive functions, the only primitive function with a pole at $s = 1$ is the Riemann zeta function $\zeta(s)$, and $F_\theta(s)$ is a primitive function if $\theta \in \mathbb{R}$ and if $F \in \mathcal{S}$ are primitive and entire (see [4, Section 4]).

We say a primitive function $F \in \mathcal{S}$ is normal if $\theta_F = 0$. Assuming Selberg orthonormality conjecture, we normalize any primitive function $F \in \mathcal{S}$ in the following way:

Received 26 May 1999.
1991 *Mathematics Subject Classification*. Primary 11M41; Secondary 11M99.
Authors' research partially supported by the State Committee for Scientific Research of Poland (KBN) grant number 2 PO3A 024 17.