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VANISHING CYCLES AND MONODROMY OF COMPLEX POLYNOMIALS

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1. Introduction. In this paper we describe the trivial summand for monodromy around a fibre of a polynomial map $\mathbb{C}^n \to \mathbb{C}$, generalising and clarifying work of Artal Bartolo, Cassou-Noguès, and Dimca [2], who proved similar results under strong restrictions on the homology of the general fibre and singularities of the other fibres. They also showed that a polynomial map $f : \mathbb{C}^2 \to \mathbb{C}$ has trivial global monodromy if and only if it is "rational of simple type" in the terminology of Miyanishi and Sugie. We refine this result and correct the Miyanishi-Sugie classification of such polynomials, pointing out that there are also nonisotrivial examples.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a nonconstant polynomial map. The polynomial describes a family of complex affine hypersurfaces $f^{-1}(c)$, $c \in \mathbb{C}$. It is well known that the family is locally trivial, so the hypersurfaces have constant topology, except at finitely many *irregular* fibres $f^{-1}(c_i)$, $i = 1, \dots, m$ whose topology may differ from the generic or *regular* fibre of f.

Definition 1.1. If $f^{-1}(c)$ is a fibre of $f : \mathbb{C}^n \to \mathbb{C}$, choose ϵ sufficiently small that all fibres $f^{-1}(c')$ with $c' \in D^2_\epsilon(c) - \{c\}$ are regular and let $N(c) := f^{-1}(D^2_\epsilon(c))$. Let $F = f^{-1}(c')$ be a regular fibre in $N(c)$. Then

$$V_q(c) := \operatorname{Ker}\bigl(H_q(F;\mathbb{Z}) \longrightarrow H_q(N(c);\mathbb{Z})\bigr)$$
$$V^q(c) := \operatorname{Cok}\bigl(H^q(N(c);\mathbb{Z}) \longrightarrow H^q(F;\mathbb{Z})\bigr)$$

are the groups of *vanishing q-cycles* and *vanishing q-cocycles* for $f^{-1}(c)$. They have the same rank, which we call the *number of vanishing q-cycles* for $f^{-1}(c)$.

Choose a regular value c_0 for f and paths γ_i from c_0 to c_i for $i = 1, \dots, m$, which are disjoint except at c_0. We can use these paths to refer homology or cohomology of a regular fibre near one of the irregular fibres $f^{-1}(c_i)$ to the homology or cohomology of the reference regular fibre $F = f^{-1}(c_0)$.

The fundamental group $\Pi = \pi_1\bigl(\mathbb{C} - \{c_1, \dots, c_m\}\bigr)$ acts on the homology $H_*(F;\mathbb{Z})$ and cohomology $H^*(F;\mathbb{Z})$. If this action is trivial, we say that f has *trivial global monodromy group*. This action has the following generators.

Let $h_q(c_i) : H_q(F) \to H_q(F)$ and $h^q(c_i) : H^q(F) \to H^q(F)$ be the monodromy

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