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Maximum Principle for Analytic Functions on Open Riemann Surfaces.

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1. Let $\mathfrak{F}$ be a non-compact region on an open Riemann surface F, such that its relative boundary Γ_0 consists of a finite number of closed analytic curves on F. Now let $w(P)$ be a single-valued analytic function on $\mathfrak{F}$, satisfying a condition

$$\text{(1)} \qquad \overline{\lim_{\Gamma_0}} \mid w(P) \mid \leqq 1.$$

We consider an arbitrary compact ring domain $G \subset \mathfrak{F}$, whose boundary consists of Γ_0 and Γ, where Γ is composed of a finite number of closed analytic curves and separates Γ_0 from the ideal boundary $\mathfrak{J}$ of $\mathfrak{F}$. If we put

$$\operatorname*{Max}_{P \epsilon \Gamma} \mid w(P) \mid \equiv M(\Gamma),$$

then we have

$$\text{(2)} \qquad \log|w(P)| \leqq \omega(P,\Gamma,G) \log M(\Gamma), \text{ for } P \epsilon G,$$

where $\omega_G(P) \equiv \omega(P, \Gamma, G)$ denotes the harmonic measure of Γ with respect to G. Namely, since $\omega(P, \Gamma, G) \log M(\Gamma) - \log |w(P)|$ is single-valued, harmonic in $P \epsilon G-S$ (where $S=E\{P;\ w(P)=0,\ P \epsilon G+\Gamma_0+\Gamma\}$) and $\geqq 0$ for P on Γ_0, Γ and arbitrarily large in the neighborhood of S, hence we easily obtain (2) by use of the maximum principle for harmonic function in compact region.

2. We fix an arbitrary point $P_0 \epsilon G$ and consider the level curve Γ^G: $\omega_G(P) = \omega_G(P_0)$. Then Γ^G consists of a finite number of closed analytic curves (occasionally with multiple points) on G and separates Γ_0 from Γ. Clearly it contains a curve passing through P_0. In following we shall denote the ring domain (on $\mathfrak{F}$) by $R(\Gamma,\Gamma')$ which is surrounded by two disjoint arbitrary boundaries Γ and Γ'. Let $R(\Gamma_0, \Gamma^G) \equiv G^*$, where Γ^G is homologous to Γ_0, then

$$\omega_G^* \equiv \omega(P, \Gamma, G) / \omega_G(P_0)$$

is clearly the harmonic measure Γ^G with respect to G^* and its