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Note on intersection multiplicity of proper components of algebraic or algebroid varieties

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Let $\mathfrak{o}$ be the ring of polynomials or the ring of formal power series in indeterminates $x_1, \cdots, x_n$ over a field k. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in $\mathfrak{o}$ and let $\mathfrak{n}$ be a minimal prime divisor of $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}$. It is easy to see that rank $\mathfrak{n} \leq$ rank $\mathfrak{p}+$ rank $\mathfrak{q}$.[1] When rank $\mathfrak{n}=$ rank $\mathfrak{p}+$rank $\mathfrak{q}$, we say that $\mathfrak{n}$ is a proper component of $\mathfrak{p} \cup \mathfrak{q}$. On the other hand, the multiplicity $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q})$ of a minimal prime divisor $\mathfrak{n}$ of $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}$ with respect to $\mathfrak{p} \cup \mathfrak{q}$ is defined as follows: Let $\mathfrak{o}'$ be a copy of $\mathfrak{o}$ and we construct $\mathfrak{o}^* = \mathfrak{o} \times_k \mathfrak{o}'$.[2] We denote by $\mathfrak{d}$ the set $\{x_1 - x_1', \cdots, x_n - x_n'\}$, where x_i' is the copy of x_i (in $\mathfrak{o}'$). Let $\mathfrak{q}'$ be the copy of $\mathfrak{q}$. Set $\mathfrak{n}^* = (\mathfrak{n}, \mathfrak{d})\mathfrak{o}$. It is evident that $\mathfrak{n}^*$ is a prime ideal of $\mathfrak{o}^*$. Set $\hat{\mathfrak{o}} = \mathfrak{o}^*_{\mathfrak{n}^*}$. Then we define

$$i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) = e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}).$$ [3][4]

The purpose of the present paper is to show the following

Theorem. Assume that $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ is a prime ideal of $\hat{\mathfrak{o}}$ and that $\mathfrak{n}$ is a proper component of $\mathfrak{p} \cup \mathfrak{q}$. Then we have

(1) $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) \leqq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only if $\mathfrak{p}\mathfrak{o}_{\mathfrak{n}}$ is generated by elements of number rank $\mathfrak{p}$;

(2) $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) \leq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only

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1) It is easy to see that if $\mathfrak{r}$ is a regular local ring and if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of $\mathfrak{r}$, then for any minimal prime divisor $\mathfrak{n}$ of $(\mathfrak{p}, \mathfrak{q})\mathfrak{r}$ we have rank $\mathfrak{n} \leqq$ rank $\mathfrak{p}+$rank $\mathfrak{q}$.

2) When $\mathfrak{o}$ is the ring of polynomials, we mean under this notation $\mathfrak{o} \times_k \mathfrak{o}'$ the tensor product of $\mathfrak{o}$ and $\mathfrak{o}'$ over k (therefore $\mathfrak{o} \times_k \mathfrak{o}' = k[x_1, \cdots, x_n, x_1', \cdots, x_n']$); when $\mathfrak{o}$ is the ring of formal power series, we mean under the same notation the Kroneckerian product of $\mathfrak{o}$ and $\mathfrak{o}'$ over k in the sense of C. Chevelley, Intersections of algebraic and algebroid varieties, Trans. Amer. Math. Soc. 57 (1945), pp. 1-85 (in this case, $\mathfrak{o} \times_k \mathfrak{o}' = k\{x_1, \cdots, x_n, x_1', \cdots, x_n'\}$).

3) Cf. C. Chevalley, 1. c. note 2).