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Note on a paper of Samuel concerning asymptotic properties of ideals

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Previously Samuel [4] defined an equivalence relation between ideals of a Noetherian ring as follows:

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a Noetherian ring $\mathfrak{o}$ having the same radical. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are not nilpotent. For every natural number n, define the integers $v_{\mathfrak{b}}(\mathfrak{a}, n)$ and $w_{\mathfrak{b}}(\mathfrak{a}, n)$ such that[1)]

(1) $\mathfrak{a}^n \subseteq \mathfrak{b}^{v_{\mathfrak{b}}(\mathfrak{a}, n)}, \quad \mathfrak{a}^n \nsubseteq \mathfrak{b}^{v_{\mathfrak{b}}(\mathfrak{a}, n)+1}$

(2) $\mathfrak{b}^{w_{\mathfrak{b}}(\mathfrak{a}, n)} \subseteq \mathfrak{a}^n, \quad \mathfrak{b}^{w_{\mathfrak{b}}(\mathfrak{a}, n)-1} \nsubseteq \mathfrak{a}^n.$

Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent* if $\lim(v_{\mathfrak{b}}(\mathfrak{a}, n)/n) = \lim(w_{\mathfrak{b}}(\mathfrak{a}, n)/n) = 1$[2)]. He showed that this defines actually an equivalence relation and that the operations of multiplication and addition are compatible with the equivalence relation.

Concerning this equivalence relation, Muhly [1] proved that if $\mathfrak{o}$ is a Noetherian integral domain, then this equivalence relation is characterized by integral dependence. Namely, we define the integral dependence as follows: An element a is *integral* over an ideal $\mathfrak{a}$ if there are elements $c_1, c_2, \cdots, c_n$ such that (i) $c_i \in \mathfrak{a}^i$ and (ii) $a^n + c_1 a^{n-1} + c_2 a^{n-2} + \cdots + c_n = 0$; an ideal $\mathfrak{b}$ is integally dependent on $\mathfrak{a}$ if every element of $\mathfrak{b}$ is integral over $\mathfrak{a}$. Then Muhly obtained the result: Two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Noetherian integral domain are equivalent to each other if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are integrally dependent on each other.

We shall prove at first that *the equivalence relation is characterized by integral dependence* without assuming that the ring is an integral domain (a generalization of the Muhly's result).

The second problem. Samuel [4] proved the following "Cancellation law":