MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXI, Mathematics No. 1, 1958.

# Some Basic Theorems in Partial Differential Algebra

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(Communicated by Prof. Akizuki, September 1, 1957)

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**Introduction.** Let $F$ be an arbitrary partial differential field of characteristic 0 with $m$ types of differentiation $\delta_1, \cdots, \delta_m$, $m \geq 0$, and let $u$ be an element from an extension field $K$ of $F$. In 1934, Raudenbush [4] defined $u$ to be algebraic over $F$ if $n$ together with its derivatives satisfy some non-trivial polynomial relation over $F$. There are, of course, the three well-known to verify:

Axiom 1. *$u_i$ is algebraic over $F<u_1, \cdots, u_n>$, $i=1, \cdots, n$.*

Axiom 2. *If $u$ is algebraic over $F<v>$ but not over $F$, then $v$ is algebraic over $F<u>$.*

Axiom 3. *If $v$ is algebraic over $F<u_1, \cdots, u_n>$ and each $u_i$ is algebraic over $F$, then $v$ is algebraic over $F$.*

Axiom 1 is trivial; and Axiom 2 also follows easily. Axiom 3, however, required a straightforward but rather complicated computational argument. The Steinitz theory of transcendency was thus established. One may say that Raudenbush's definition (to be referred to as *Definition* I) is adapted to Axiom 2, but not to Axiom 3. On the other hand, consider the following definition. By the $\delta_i$-theory we mean the theory which results from regarding $K$ as a partial differential field under the $m-1$ differentiations $\delta_1, \cdots, \delta_{i-1}, \delta_{i+1}, \cdots, \delta_m$. The definition runs:

**Definition. II.** *We say that $u$ is algebric over $F$ (for $m>0$) if $F<u>/F$ is of finite degree of transcendency in each of the $\delta_i$-theories, $i=1, \cdots, m$. (For $m=0$, the usual definition is to obtain.)* Here Axiom 3, stated in terms of Definition II, becomes:

*If $F<u_1, \cdots, u_n, v>/F<u_1, \cdots, u_n>$ and $F<u_1, \cdots, u_n>/F$ are both of finite degree of transcendency in each of the $\delta_i$-theories,*