Brownian motions on the 3-dimensional rotation group

By

H. P. McKEAN, JR.

1. INTRODUCTION

K. Itô [1950]* and K. Yosida [1952] defined and constructed all the Brownian motions on Lie groups. The purpose of the present paper is to give a new method for constructing the Brownian sample paths on the 3-dimensional rotation group SO (3). The idea is to inject the differentials $\mathfrak{z}(dt)$ of a (skew) Brownian motion on the Lie algebra into SO (3) via the exponential map e and to piece the resulting infinitessimal rotations $e[\mathfrak{z}(dt)]$ into a continuous path (product integral):

1.1 $$\mathfrak{g}_\infty(t)$$
$$= \bigwedge_{s \leq t} e[\mathfrak{z}(ds)]$$
$$= \lim_{n \uparrow \infty} e[\mathfrak{z}(0, 2^{-n})] \cdots e[\mathfrak{z}(2^{-n}[2^n t], t)] \qquad t \geq 0 .$$

The same trick gives the Brownian motions on all the classical (non-exceptional) simple Lie groups of É. Cartan's list.

F. Perrin [1928] computed the counterparts of the Gauss and Poisson laws on SO (3); for a sketch of Perrin's results, see P. Lévy [1948: 194-203].

I divide the paper into 8 sections: 2 deals with SO (3), its Lie algebra, and its differential operators; 3 with its Brownian motions. 4 states the program of injection. 5 is devoted to sums $\mathfrak{g} = \sum_{n \geq 0} \mathfrak{j}_n$ of stochastic integrals

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* K. Itô [1950] means K. Itô's 1950 publication listed at the end of this paper; K. Itô [1950: 6-8] would mean pages 6-8 of that work.