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Note on direct summands of modules

By

Takehiko MIYATA

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Some years ago the following question was raised by H. Matsumura and solved affirmatively by H. Toda:

"Let N be a subgroup of a finitely generated abelian group M. Suppose that M and $N\oplus M/N$ are isomorphic, is N a direct summand of M?"

A simple proof runs as follows. Let n be an arbitrary integer, M/nM is isomorphic to $N/nN\oplus((M/N)/n(M/N))$. If we denote the order of a finite group G by Card (G), then

$$\begin{aligned}\text{Card}(N/nN) &= \text{Card}(M/nM)-\text{Card}((M/N)/n(M/N))\\ &= \text{Card}(M/nM)-\text{Card}(M/(N+nM))\\ &= \text{Card}(nM/(N+nM))\\ &= \text{Card}(N/(N\cap nM))\end{aligned}$$

Therefore $nN=N\cap nM$ for an arbitrary integer, that is, N is a *pure subgroup* of M. By a well known theorem N is a direct summand of M.

In this note we show that this property holds for more general class of modules. The notion of pure subgroups may be generalized to that of modules (Exercise 24, Chap. 1, §2 [1]). We will prove in this context the analogy of the classical theorem that a pure subgroup N of an abelian group M is a direct summand of M provided that M/N is finitely generated.

First we list more or less well known lemmas without proofs. Throughout this note R is a Noetherian commutative ring and A an R-algebra which is of finite type as an R-module. When R