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# Rings with nonzero singular ideals

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In recent years, there have been many results about rings and their various types of ring of quotients including the case of classical quotient ring. However all significant results are so far limited to the case where the singular ideal is identically zero. The difficulty in the general case lies in the nonclosed property of the singular ideal. In this paper we study some of the properties of the closure of the singular ideal of a ring $R$ and the relations between the rings of quotients of $R$ and the rings of quotients of factor rings of $R$.

Let $R$ be a ring with identity element 1. If $S$ is a subset of $R$ and $x\in R$, we denote $x^{-1}(S)=\{r\in R\mid xr\in S\}$. We also denote the right and left annihilators of $S$ in $R$ by $\gamma(S)$ and $l(S)$ respectively. The singular ideal (right singular ideal) $J(R)$ of $R$ is defined as $J(R)=\{r\in R\mid \gamma(r)$ is an essential right ideal of $R\}$. The closure $K(R)$ of $J(R)$ is defined as $K(R)=\{k\in R\mid k^{-1}(J(R))$ is an essential right ideal of $R\}$. $K(R)$ is a two sided ideal in $R$ and is the unique maximal essential extension of $J(R)$ in $R$ as right $R$-module. Let $\tilde{R}=R/J(R)$. Since the inverse image of an essential right ideal in $\tilde{R}$ is essential in $R$. $\widetilde{K(R)}$, the image of $K(R)$ in $\tilde{R}$, contains the singular ideal $J(\tilde{R})$ of $\tilde{R}$. It is not true that they are equal always. In the case where $J(R)$ is essential in $R$, $K(R)=R$ whereas $J(\tilde{R})\neq\tilde{R}$.

**Lemma 1.1.** *The following statements are equivalent*

1. $\widetilde{K(R)}=J(\tilde{R})$
2. $k_1, k_2\in K(R)$ *there exists* $r\in R$ *such that* $(k_1k_2)r\in J(R)$, $k_2r\notin J(R)$ *if* $k_2\notin J(R)$.