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Faithfully flatness of extensions of a commutative ring

By

Takeo OHI

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All rings are assumed to be commutative and contain an identity element. Moreover, when we write "$R \subseteqq S$", we mean that R is a subring of a ring S and that the identity of R is the identity of S. Let $a_1, \dots, a_s$ be elements of a ring R. Then $(a_1, \dots, a_s)$ means the ideal of R generated by $a_1, \dots, a_s$. The symbol $\subset$ means proper inclusion.

Let R and S be rings with $R \subseteqq S$. If S is generated as a ring by a set of elements in S over R, then S may be best described by an exact sequence of R-homomorphisms

$$(*) \qquad 0 \longrightarrow I \longrightarrow R[X] \longrightarrow S \longrightarrow 0$$

where X is a set of variables over R and $R[X]$ is the polynomial ring in X over R. If $w = r_0 + r_1 X^{(1)} + r_2 X^{(2)} + \cdots + r_n X^{(n)}$ is an element of $R[X]$ where $r_0, r_1, \dots, r_n \in R$ and $X^{(i)}$ monomials with degree $\geqq 1$ such that $X^{(i)} \neq X^{(j)}$ if $i \neq j$, then we define $c(w)$ (and $c'(w)$) to be $(r_0, r_1, \dots, r_n)$ (and $(r_1, \dots, r_n)$).

We say that an R-module M is faithfully flat if M is flat over R and $PM \subset M$ for any maximal ideal P of R. Let the notations be as above. In this note we seek some conditions which are necessary and sufficient in order that S is, as an R-module, faithfully flat. They will be characterized in terms of the R-module I, which is also the ideal in $R[X]$.

In order to prove the Theorem 1 we need the following two Lemmas. Lemmas A and B may be found in [1, Chapter 1, § 3,