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Growth properties of solutions of second order elliptic differential equations

By

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Introduction

In this paper we consider the equation

$$(1)\qquad -\sum_{j,k=1}^{n} D_j a_{jk}(x) D_k u - q(x)u + p(x)u = 0$$

in an exterior domain $\Omega \subset \mathbf{R}^n$, where $D_j = \partial_j + ib_j(x)$ with $\partial_j = \partial/\partial x_j$ and $i = \sqrt{-1}$, and the matrix $(a_{jk}(x))$ is uniformly positive definite in $x \in \Omega$ (the precise condition on the coefficients will be given later). We assume that $a_{jk}(x) \to \delta_{jk}$ (Kronecker's delta) as $|x| \to \infty$, that $\partial_k b_j(x) - \partial_j b_k(x)$ and $p(x)$ behave like $o(|x|^{-1})$ as $|x| \to \infty$ and that there exist some constants $0 < \gamma_0 < 1$, $\lambda_0 > 0$ and $r_0 > 0$ such that the domain $B(r_0) = \{x; |x| > r_0\}$ is included in Ω and

$$(2)\qquad 2\gamma_0 \Big(\sum_{j,k} a_{jk}(x) \tilde{x}_j \tilde{x}_k\Big) q(x) + |x| \sum_{j,k} \tilde{x}_j a_{jk}(x) \partial_k q(x) \geq \lambda_0 \quad \text{for} \quad x \in B(r_0),$$

where $\tilde{x} = x/|x|$. The main purpose of the present paper is to derive a growth estimate at infinity of solutions $u(x)$ of equation (1), from which will follow the uniqueness of L^2-solutions of (1).

Equations of the form (1) appear frequently in applications. In particular, if we assume that $a_{jk}(x) = \delta_{jk}$ and $q(x) = \lambda - c(x)$, where $\lambda > 0$, then (1) becomes

$$(3)\qquad -\sum_{j=1}^{n} D_j^2 u + (c(x) + p(x))u - \lambda u = 0,$$