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Good and bad field generators

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Let k be a field. A *field generator* in two variables over k is a polynomial $f \in k[x, y]$ such that $k(x, y)=k(f, g)$ for some rational function $g \in k(x, y)$. We continue the investigation of field generators begun in [1] and [2]. Using methods of [2], we first study in detail properties of the multiplicity tree at infinity of f once coordinate functions x, y have been chosen that are natural for f (see [2, 4.7]). Our original motivation for this had been an attempt to show that all field generators are *good* in the sense that a complementary generator g can be found in $k[x, y]$. However, a quite astonishing example of a bad field generator has been constructed by C. Jan in [1], and we instead use the numerical information obtained to determine, with the help of a computer, all bad field generators of degree ≤ 25, the degree of Jan's example. We find that field generators are good for degrees $d \leq 20$ and $d=22, 23, 24$, and that there is exactly one "type" of bad field generator for $d=21$ and $d=25$ (see 2.6 for a more precise statement). R. Ganong helped materially with the rather elaborate calculations needed to establish this and with the writing of an appendix in which some of the details are explained.

A good field generator f appears as part of a birational morphism φ: $\mathbf{A}_k^2 \rightarrow \mathbf{A}_k^2$ with $\varphi(\alpha, \beta)=(f(\alpha, \beta), g(\alpha, \beta))$ for $\alpha, \beta \in k$. We show that this is almost true in general. Namely, if f is a field generator, a complementary generator $g=a/b$ can always be found with $a, b \in k[x, y]$ such that $(a, b)k[x, y]=k[x, y]$. This means that the pencil of curves $\{g-\mu \mid \mu \in k\}$ has no base points at finite distance and that φ: $\mathbf{A}_k^2 \rightarrow \mathbf{P}_k^2$, $\varphi(\alpha, \beta)=(1, f(\alpha, \beta), g(\alpha, \beta))$, is a birational morphism.

1. We assume that k is algebraically closed in the sequel. This is done mainly to simplify arguments and could be avoided in most places. We use systematically the notation of [2]. Also, if S is a non-singular surface and $p \in S$, π_p: $S' \rightarrow S$ will denote the locally quadratic transformation (l.q.t.) with centre p and $E_p=\pi_p^{-1}(p)$ its exceptional fibre. E_0 will stand for the line at