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On submodules of a Verma module The case of $\mathfrak{sl}(4, \mathbf{C})$

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Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the root system of $(\mathfrak{g}, \mathfrak{h})$. Denote by $\mathfrak{g}^\alpha$ the root space corresponding to a root α, then $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$. We fix a positive system of roots Δ_+ and denote by Δ_0 the set of simple roots. Put

$$\mathfrak{n}^+=\sum_{\alpha\in\Delta_+}\mathfrak{g}^\alpha,\ \mathfrak{n}=\sum_{\alpha\in-\Delta_+}\mathfrak{g}^\alpha,\ \rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha.$$

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. For any $\chi\in\mathfrak{h}^*=\mathrm{Hom}(\mathfrak{h}, \mathbf{C})$, we consider the factor space $M(\chi)=U(\mathfrak{g})/I_\chi$, where I_χ is the left ideal of $U(\mathfrak{g})$ generated by $\mathfrak{n}^+$ and $\{H-\chi(H)+\rho(H);\ H\in\mathfrak{h}\}$. Then $M(\chi)$ has the natural structure of $U(\mathfrak{g})$-module and is called the Verma module induced by χ. A nonzero element of a $U(\mathfrak{g})$-module is called extreme if it is annihilated by $\mathfrak{n}^+$.

D.-N. Verma proved in [1] that a submodule of $M(\chi)$ generated by its extreme vector is isomorphic to another Verma module $M(\chi')$. The submodules of this type are called here Verma submodules. He also got a sufficient condition on a pair (χ, χ') for $M(\chi)$ to contain a Verma submodule isomorphic to $M(\chi')$.

After that, I. N. Bernstein and others proved that this condition is also necessary [2]. So all the Verma submodules are already known.

In that work [2], they also constructed an example of submodules which are not generated by their extreme vectors. They treat there the case $\mathfrak{g}=\mathfrak{sl}(4, \mathbf{C})$ and χ is a certain weight ω (see § 2). J. Dixmier and N. Conze gave a fundamental necessary condition for the existence of submodules which are not of Verma's type.

It is an interesting problem to determine the structure of the Verma module $M(\chi)$, and especially to find the submodules of $M(\chi)$, not of Verma's type.