J. Math. Kyoto Univ. (JMKYAZ)
17-3 (1977) 533-538

Complete intersections

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(Communicated by Prof. M. Nagata, May 22, 1976)
(Revised, Jan. 14, 1977)

In [2], D. Ferrand has given some characterisation of a reduced scheme X which is a local complete intersection, in terms of $\Omega_X{}^1$ [the sheaf of 1-differentials]. In the global affine case, Murthy and Towber [3] have proved that a smooth affine curve over an algebraically closed field is a complete intersection in any embedding of it in an affine space if only and if the module of 1-differentials of the curve is trivial. It is not known whether there exist any intrinsic properties of an affine shceme, which will determine whether it is a complete intersection in any embedding of it in an affine space. Here we prove the following:

Let R be a finite type k-algebra which is a domain, where k is any field and the quotient field of R is separable over k. Then R is a complete intersection in some embedding of it in an affine space over k if and only if the module of 1-differentials, $\Omega^1_{R/k}$, has a free resolution of length $\leqslant 1$. We also prove that when R is smooth over k, for embeddings in large dimensional affine spaces it is a complete intersection, if it is so in some embedding. As a corollary we deduce that the conormal bundle of a local complete intersection in any embedding, is a complete intersection in some embedding. Finally we give examples of smooth affine varieties which have trivial canonical line bundles, but not a complete intersection in any embedding of it in affine space, thereby settling a question of M. P. Murthy [6].

I wish to thank C. J. Rego and R. C. Cowsik for helping and encouraging me through many discussions. Thanks are due to M. V. Nori for the final remark.

We will first prove an elementary lemma which is the key lemma.

Lemma. *Let R be a commutative ring with unity and I a finitely generated ideal of R. Let I/I^2 be generated by r elements as an R/I-module. Let F be any element of R. Then the ideal $(I, F)\subset R$ is generated by $r+1$ elements.*

Proof. Let $a_1, \cdots, a_r$ be elements of I such that their residues mod I^2 generate I/I^2. So in the ring $R/(a_1, \cdots, a_r)R$, the ideal $\bar{I}=I/(a_1, \cdots, a_r)R$ has the property that $\bar{I}/\bar{I}^2=0$. i.e. $\bar{I}=\bar{I}^2$. Since $\bar{I}$ is finitely generated, we see that $\bar{I}$ is generated by an idempotent. Let $h\in I$ be any lift of this idempotent in $\bar{I}$.