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On homotopy equivalences of $S^2 \times RP^2$ to itself

By

Takao MATUMOTO

Dedicated to Professor A. Komatu on his 70th birthday

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In this note the general exact sequence method to calculate the based homotopy set $[X, Y]_0$ is presented, including the case in which Y is not simple (§ 0). As an application, we shall determine the based homotopy set $[S^2 \times RP^2, S^2 \times RP^2]_0$ and the group $\mathcal{E}(S^2 \times RP^2)$ of self homotopy equivalences, where S^2 and RP^2 denote the 2-sphere and the real projective plane respectively (Theorem 1, Lemmas 1.1 and 1.4).

Which homotopy classes are representable by diffeomorphisms (Corollary 2.1)? This question leads us to study the homotopy smoothings from the surgery theoretical point of view. We shall show that any homotopy smoothing of $S^2 \times RP^2$ is s-cobordant to a homotopy equivalence of $S^2 \times RP^2$ to itself (Corollary 2.2). Similarly, any smooth s-cobordism of $S^1 \times RP^2$ to itself is shown to be s-cobordant to the product cobordism $S^1 \times RP^2 \times I$ relative to the boundary (Proposition 3). We refer [11] for the topological s-cobordisms.

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§ 0. Generalities on the based homotopy set $[X, Y]_0$

Let X and Y be connected CW complexes with based point. We are concerned with the space $Map_0(X, Y)$ of based point preserving maps of X into Y equipped with compact-open topology. In order to study the set of based homotopy classes $\pi_0(Map_0(X, Y))$, or in a more familiar notation $[X, Y]_0$, we can use two types of filtrations.

The first one comes from the structure of a CW complex with unique 0-cell having the same homotopy type as X. Let X^n denote the n-skeleton of this CW complex. Then, we have the Puppe cofibering sequence,

$$S^{n-1} \vee \cdots \vee S^{n-1} \xrightarrow{f} X^{n-1} \xrightarrow{j} Cf = X^n \longrightarrow S^n \vee \cdots \vee S^n \xrightarrow{S(f)} SX^{n-1} \longrightarrow \cdots$$

and the following induced exact sequence in the category of pointed sets.