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Eikonal equations and spectral representations for long-range Schrödinger Hamiltonians

By

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§1. Introduction

We shall investigate in this paper spectral representations for the Schrödinger operator defined as the self-adjoint realization in $L_2(\mathbf{R}^n)$ of $H=-\Delta+V(x)$, where Δ denotes the Laplacian in $L_2(\mathbf{R}^n)$ and the *potential* $V(x)$ satisfies the following

Assumption: *$V(x)$ is a real-valued $C^3(\mathbf{R}^n)$-function such that for some $\delta>0$*

$$D_x^\alpha V(x)=O(r^{-|\alpha|-\delta}) \quad as \quad r=|x|\to\infty \quad (0\le|\alpha|\le 3)$$

where $D_x=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$, α is a multi-index.

As has been noted by Ikebe [2], the usual Fourier transform (precisely speaking, its restriction to the sphere) is obtained from the asymptotic expansion of the solution of the Helmholtz equation (for instance in $\mathbf{R}^3$, and if $f\in C_0^\infty(\mathbf{R}^3)$,

$$(1.1) \qquad \frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}f(y)dy=\frac{e^{i\sqrt{\lambda}r}}{4\pi r}\int_{\boldsymbol{R}^3}e^{-i\sqrt{\lambda}\omega y}f(y)dy+O(r^{-2})$$

as $r=|x|\to\infty$, where $\omega=x/r$).

Suggested by the above observation, Ikebe [2] and Saitō [10] have obtained the spectral representation theorems for Schrödinger operators with long-range potentials by considering the following limit

$$(1.2) \qquad \lim_{r\to\infty} r^{(n-1)/2}e^{-iK(x,\lambda)}(R(\lambda+i0)f)(r\cdot)$$

in $L_2(S^{n-1})$, where $R(z)=(H-z)^{-1}$, and it has been observed that $K(x,\lambda)$ should be chosen as an (approximate) solution of the *eikonal equation*

$$(1.3) \qquad |\nabla_x K(x,\lambda)|^2+V(x)=\lambda$$

(see Ikebe-Isozaki [3]). This procedure has also been adopted by Mochizuki-

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