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On a lattice property of the space $\Gamma_{ho} \cap \Gamma_{he}$

By

Fumio MAITANI

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Introduction

In the Hilbert space $\Gamma = \Gamma(R)$ of square integrable real differentials on a Riemann surface R, let Γ_h (resp. Γ_{he}) be the subspace of Γ which consists of harmonic (resp. exact harmonic) differentials. The orthogonal complement of Γ_{he}^* in Γ_h is denoted by Γ_{ho}. The space of harmonic measure Γ_{hm} is defined as follows: $\omega \in \Gamma_{hm}$ if and only if for every $\varepsilon > 0$ and every compact set E there exist a canonical region G $(\supset E)$ and a harmonic function w_G which is constant on each boundary component of G such that $\|\omega - \mathrm{d}w_G\|_G < \varepsilon$. The subspace Γ_{hm} is the orthogonal complement of the space Γ_{hse}^* in Γ_h, where Γ_{hse} consists of harmonic semiexact differentials. The subspace $\Gamma_{ho} \cap \Gamma_{he}$ clearly includes Γ_{hm}, and $\Gamma_{ho} \cap \Gamma_{he} = \Gamma_{hm}$ for finite bordered Riemann surfaces, but R. Accola showed an example of Riemann surface of infinite genus for which the equality does not hold.

Now for HP-functions u and v we denote by $u \wedge v$ (resp. $u \vee v$) the greatest harmonic minorant (resp. the least harmonic majorant) of u and v. A subspace $\Gamma_x \subset \Gamma_{he}$ forms a vector lattice if $\mathrm{d}u$ and $\mathrm{d}v \in \Gamma_x$ imply $\mathrm{d}(u \wedge v)$ and $\mathrm{d}(u \vee v) \in \Gamma_x$. We say that a subspace $\Gamma_x \subset \Gamma_{he}$ has a lattice property if $\mathrm{d}f \in \Gamma_x$ implies $\mathrm{d}(f \wedge c) \in \Gamma_x$ for every real constant c. The space Γ_{he} forms the vector lattice, hence it has a lattice property. It is pointed out in [4] that the space Γ_{hm} has the lattice property.

Here we shall show that $\Gamma_{ho} \cap \Gamma_{he}$ has also the lattice property. Some related subjects shall be investigated.

1. We shall show first that Γ_{hm} forms the vector lattice. This implies that Γ_{hm} has the lattice property.

Proposition 1. *Let u and v be harmonic functions such that* $\mathrm{d}u$ *and* $\mathrm{d}v$ *belong to Γ_{hm}. Then* $\mathrm{d}(u \wedge v)$ *and* $\mathrm{d}(u \vee v)$ *belong to Γ_{hm}.*

Proof. It is sufficient to show $\mathrm{d}(u \wedge v) \in \Gamma_{hm}$. Let $\{R_n\}$ be a canonical regular exhaustion of R. For a given $\varepsilon > 0$, there exists an R_m such that $\|\mathrm{d}u\|_{R-R_m} < \varepsilon$ and $\|\mathrm{d}v\|_{R-R_m} < \varepsilon$. Further, there exist an integer N $(>m)$ and harmonic functions u_n and v_n in R_n $(n > N)$ such that u_n and v_n are constant on each boundary component of R_n, $\|\mathrm{d}(u - u_n)\|_{R_n} < \varepsilon$, and $\|\mathrm{d}(v - v_n)\|_{R_n} < \varepsilon$. We have a continuous extension $\hat{u}_n$