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On ideal-adic completion of noetherian rings

By

Jun-ichi NISHIMURA

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Introduction

In commutative (noetherian) ring theory, complete local rings play many important roles. Thanks to the efforts made by Krull, Zariski, Nagata and Grothendieck, a lot of marvelous properties of complete local rings are known. Moreover, they applied the knowledge to investigate the (local) properties of general noetherian rings, using the maximal ideal-adic completion.

In particular, discovering many beautiful properties of complete local rings, Nagata successfully used them in the investigation, for example, of the finiteness problem of integral closures of noetherian domains. In this work, he found an acceptable class of noetherian rings which possess the (universal) finiteness property for integral closures. He named these rings "pseudo-geometric" (for definition, see [7, (31.A)]). We note here that he found the examples of bad local rings at the same time (cf. [8, Appendix]).

In reconstructing Nagata's work, Grothendieck noticed the importance of the informations included in formal fibres, which connect a local ring with its completion. Developing the concept of formal smoothness, he paid a special attention to the study of noetherian rings whose formal fibres are geometrically regular. He also found a new class of noetherian rings which have algebraic-geometrically reasonable properties. He called them "excellent" (for definition, see [7, (34.A)]).

Since complete local rings are proved to be always excellent, Grothendieck expected that the situation of formal fibres of noetherian rings may become better when one completes the noetherian rings in an ideal-adic topology. He asked if, for a noetherian ring A having good formal fibres and for an (arbitrary) ideal I of A, the completion A^* of A in the I-adic topology has also good formal fibres. More precisely, letting $\boldsymbol{P}$ denote a certain (ring-theoretic) condition, Grothendieck defined a $\boldsymbol{P}$-ring to be a noetherian ring whose formal fibres satisfy the condition. In this terminology, he stated the questions as follows (cf. [3, (7.4.8)]):

Question 1. Let A be a noetherian ring and I an ideal of A. If A is a $\boldsymbol{P}$-ring, is the I-adic completion A^* of A also a $\boldsymbol{P}$-ring?

More generally