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Diffusion processes in population genetics

By

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§ 1. Introduction

In population genetics theory we often encounter diffusion processes on the compact domain $K=\{(x_1,\dots, x_d)\in R^d;\ x_1\geq 0,\dots, x_d\geq 0,\ 1-x_1-\cdots-x_d\geq 0\}$. In order to construct such diffusion processes, we will consider a martingale problem on K.

Let A be a second order differential operator on K

(1.1) $$A=\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

with domain $D(A)=C^2(K)$,[1)] where $\{a_{ij}(x)\}_{1\leq i,j\leq d}$ is a real symmetric and non-negative definite matrix defined on K and $\{b_i(x)\}_{1\leq i\leq d}$ is an R^d-valued measurable function defined on K.

We assume that $\{a_{ij}(x)\}$ and $\{b_i(x)\}$ are continuous on K. Let $\Omega=C([0,\infty):K)$ be the space of all K-valued continuous functions defined on $[0,\infty)$. For each $\omega\in\Omega$ and each $t\geq 0$, we denote $x(t:\omega)=\omega(t)$. Let $\mathscr{F}_t$ and $\mathscr{F}$ be the σ-fields generated by $\{x(s);\ 0\leq s\leq t\}$ and $\{x(s);\ s\geq 0\}$ respectively.

Let $x\in K$. A probability measure P on $(\Omega,\mathscr{F})$ is called a solution of the (K, A, x)-martingale problem if it satisfies the following conditions,

(1.2) $P[\omega;\ x(0:\omega)=x]=1$, and

(1.3) denoting $M_f(t)=f(x(t))-\int_0^t Af(x(s))ds$, $(M_f(t),\mathscr{F}_t)$ is a P-martingale for each $f\in C^2(K)$.

It is known that if a solution of the (K, A, x)-martingale problem exists, the following conditions must be satisfied, (cf. Okada [9]).

(1.4) $$a_{ii}(x)=0 \quad\text{if}\quad x_i=0,\quad\text{and}\quad \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)=0\quad\text{if}\quad \sum_{i=1}^{d} x_i=1,$$

and

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1) Each element of $C^2(K)$ is a C^2- function defined on an open set containing K.