On higher derivatives of distribution functions of class $L$

By

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1. Introduction

Zolotarev [6] and Wolfe [5] study the order of smoothness of a distribution function $F(x)$ of class $L$ on the real line. A number $N$ (a nonnegative integer or $+\infty$) is explicitly determined from Lévy’s representation of the characteristic function $\phi(t)$ of $F(x)$, so that $F(x)$ is of class $C^N$ on $(-\infty, \infty)$ and, in case $N < \infty$, $F(x)$ is not of class $C^{N+1}$ on $(-\infty, \infty)$. If $N < \infty$, then there is a point $\gamma_0$ such that $F(x)$ is of class $C^{N+1}$ on $(-\infty, \gamma_0) \cup (\gamma_0, \infty)$. The point $\gamma_0$ is also explicitly given. One of the problems we treat in [3] is to analyze asymptotic behavior of $F^{(N+1)}(x)$ as $x \to \gamma_0$. A complete description of it is obtained in the case of one-sided distributions. The two-sided case with $N < \infty$ is divided into five cases by two real numbers $\lambda_+$ and $\lambda_-$ determined from the density of the Lévy measure. If none of $\lambda_+$, $\lambda_-$, and $\lambda_+ + \lambda_-$ is an integer, then we have Case (a). In this sense the cases other than (a) are exceptional. We give in [3] the asymptotic behavior in three cases including Case (a). In particular, if $N = 0$, then the asymptotic behavior of $F'(x)$ as $x \to \gamma_0$ is fully described. The purpose of the present paper is to examine the remaining two cases. We will give several results and show that the situation is widely different from the three cases studied in [3]. The behavior is delicate and the complete description of it remains still open. As Lévy’s representation of infinitely divisible distributions has probabilistic meanings connected with stochastic processes with independent increments, it is an interesting problem to analyze in detail how the elements in the representation determine fine structures of the distribution.

The characteristic function $\phi(t)$ of a distribution function $F(x)$ of class $L$ is as follows:

$$\phi(t) = \exp \left\{ it - \frac{\sigma^2 t^2}{2} + \int_{R_0} \left( e^{itu} - 1 - \frac{itu}{1 + u^2} \right) \frac{k(u)}{u} du \right\},$$

where $\gamma$ is real, $\sigma^2 \geq 0$, $R_0 = (-\infty, 0) \cup (0, \infty)$, $k(u)$ is nonpositive on $(-\infty, 0)$ and nonnegative on $(0, \infty)$, $k(u)$ is non-increasing on each of $(-\infty, 0)$ and $(0, \infty)$, and

$$\int_{|u| \leq 1} u k(u) du + \int_{|u| > 1} \frac{k(u)}{u} du < \infty.$$

Denote $\lambda_+ = k(0+)$, $\lambda_- = |k(0-)|$, $\lambda = \lambda_+ + \lambda_-$. The number $N$ is determined as