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The characters of the discrete series for semisimple Lie groups

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Introduction

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. For a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote by $H^{\mathfrak{h}}$ the corresponding Cartan subgroup of G. Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ such that its toroidal part has the possible maximal dimension, and put $B=H^{\mathfrak{b}}$. Denote by $\mathfrak{b}_B^*$ the space of linear forms Λ of $\mathfrak{b}$ into $\boldsymbol{C}$ such that $B \ni \exp X \mapsto \exp \Lambda(X) \in \boldsymbol{C}$ $(X \in \mathfrak{b})$ defines a unitary character of B, and by $\mathfrak{b}_B^{*\prime}$ its subset consisting of regular elements. When every root of $(\mathfrak{g}_c, \mathfrak{b}_c)$ (or simply of $\mathfrak{b}$) is imaginary, we call $\mathfrak{b}$ compact. In that case, Harish-Chandra proved the existence and the uniqueness of a certain kind of invariant eigendistribution on G for $\Lambda \in \mathfrak{b}_B^{*\prime}$. When B is compact, it is characterized as a unique tempered invariant eigendistribution which coincides on $B \cap G'$ with a certain function, where G' denotes the set of all regular elements in G. We define the same kind of invariant eigendistribution π_Λ for $\Lambda \in \mathfrak{b}_B^{*\prime}$ evenwhen some roots of $\mathfrak{b}$ are not imaginary ($\pi_\Lambda = \Theta_\Lambda$ in the above case, and for the exact definition, see below).

The purpose of this paper is to give a global explicit formula of the invariant analytic function π'_Λ on G' corresponding canonically to π_Λ. We assume that G is acceptable in the sense of Harish-Chandra [2(b), § 18] for convenience. But this is not an essential restriction, and the essential assumption we made here is the connectedness of G. Our main results are Theorems 1 and 2 in § 5 which give the explicit formulas of the functions $\pi'_\Lambda(h)$ on $H^{\mathfrak{h}} \cap G'$ for every $\mathfrak{h}$.

When B is compact, G has the discrete series representations, and their characters are equal to Θ_Λ's except the known multiplicative sign ± 1. Thus we get the explicit character formula for these representations. Many researches have been made in this direction, for instance, Hecht [3], Martens [6], Midorikawa [7(a), (b)], Schmid [8(a), (b)] and Hirai [5(a), (b), (e)] (cf. also Arthur [10]). The first two authors treat essentially the holomorphic discrete series, and the next two authors treat some type (or types) of linear groups.

The method of the present paper goes along the same line as in the previous paper [5(e)]. Thus we apply the necessary and sufficient condition in [5(c)] for