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Notes on liaison and duality

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1. Introduction.

In classifying algebraic curves in projective three space Noether [8] introduced the notion of the residual intersection (liaison). Two curves $C, C' \subset \boldsymbol{P}_k^3$, k an algebraically closed field, are linked geometrically, if they have no components in common and $C \cup C'$ is a complete intersection. In his paper [10] Rao studied the following graded S-module: $M(C) = \bigoplus_{s \in \boldsymbol{Z}} H^1(\boldsymbol{P}, \mathcal{I}_C(s))$, where $S = k[X_0, \cdots, X_3]$ and $\mathcal{I}_C$ denotes the ideal sheaf of C. If C, C' are linked, then $M(C)$ and $M(C')$ are isomorphic up to duality and shifts in gradings. One of the main results of our paper is to extend Rao's result to an arbitrary Cohen-Macaulay variety $X \subset \boldsymbol{P}_k^n$. That is, we define a formal vector $\mathfrak{m}(X)$ which is shown to be an invariant up to duality and shifts in gradings under the liaison, compare 5.

Because most of our results are valid for Gorenstein ideals instead of complete intersection ideals, we consider Gorenstein liaison, compare 2.1 resp. 5.1 for the exact definition. In the context of local algebra two ideals $\mathfrak{a}$, $\mathfrak{b}$ of a local Gorenstein ring R are linked geometrically if $\mathfrak{a}$, $\mathfrak{b}$ are of pure height, have no primary components in common, and $\mathfrak{a} \cap \mathfrak{b}$ is a Gorenstein ideal. Now the question is, what kind of properties of $R/\mathfrak{a}$ can be transformed into properties of $R/\mathfrak{b}$. Even in the case of liaison with respect to complete intersection ideals most of our results are new. As a main point we define in 3. a certain complex $J_{\mathfrak{a}}^{\cdot}$, the truncated dualizing complex of $R/\mathfrak{a}$, which is shown to be an invariant (up to a shift and duality) under liaison, i.e. $J_{\mathfrak{b}}^{\cdot}$ is up to a shift and duality isomorphic to $J_{\mathfrak{a}}^{\cdot}$. As an application it follows that $R/\mathfrak{a}$ is a Cohen-Macaulay ring (locally a Cohen-Macaulay ring, resp. a Buchsbaum ring) if and only if the corresponding property holds for $R/\mathfrak{b}$. As another application of this type we show in 4.1 that the Serre condition S_r for $R/\mathfrak{a}$ is equivalent to the vanishing of the local cohomology groups $H_{\mathfrak{m}}^i(R/\mathfrak{b}) = 0$ for $\dim R/\mathfrak{b} - r < i < \dim R/\mathfrak{b}$. In the case $\mathfrak{a}$ is linked to itself that leads to a Cohen-Macaulay criterion. For this and related results compare 4. We conclude our considerations with some geometric applications, compare 5. In particular, in 5.4 we extend Rao's invariant to an arbitrary Cohen-Macaulay variety.

As a main technical tool to establish the relation between $R/\mathfrak{a}$ and $R/\mathfrak{b}$ we use