J. Math. Kyoto Univ. (JMKYAZ)
22-4 (1983) 685-699

Lie algebra of foliation preserving vector fields

By

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(Communicated by Prof. H. Toda Aug. 19, 1981)

Introduction

Let M and M' be connected paracompact C^∞-manifolds and $\mathfrak{x}(M)$ and $\mathfrak{x}(M')$ the Lie algebras of all C^∞-vector fields with compact support on M and M' respectively. A well-known theorem of Pursell-Shanks [10] may be stated as follows.

Theorem. *There exists a Lie algebra isomorphism Φ of $\mathfrak{x}(M)$ onto $\mathfrak{x}(M')$ if and only if there exists a C^∞-diffeomorphism φ of M onto M' such that $d\varphi=\Phi$.*

The above result still holds for Lie algebras of all infinitesimal automorphisms of several geometric structures on manifolds. Indeed, Omori [9] proved the corresponding result in case of volume structures, symplectic structures, contact structures and fibering structures with compact fibers. The case of complex structures was proved by Amemiya [1]. Koriyama [8] proved that this is still true for submanifolds regarding a submanifold as a geometric structure. Furthermore the first author [6] has proved the corresponding result in case of Lie algebras of G-invariant C^∞-vector fields with compact support on paracompact, connected, free G-manifolds when G is a compact connected semi-simple Lie group such that the automorphism group of its Lie algebra is connected. The corresponding result is no longer true when the automorphism group of its Lie algebra is not connected, G is not semi-simple or G does not act freely.

Let $(M, \mathscr{F})$ be a foliated manifold and $\mathfrak{x}(M, \mathscr{F})$ (resp. $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$) be the Lie algebra of all foliation preserving (resp. leaf preserving) C^∞-vector fields with compact support on M. Then we have the following theorem, due to Amemiya [1], which can be also proved by using the methods of Pursell-Shanks [10] and Omori [9].

Theorem A. *There exists a Lie algebra isomorphism Φ of $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ onto $\mathfrak{x}_{\mathscr{L}}(M', \mathscr{F}')$ if and only if there exists a foliation preserving diffeomorphism φ of M onto M' such that $d\varphi=\Phi$.*

Theorem A implies that if $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$ is algebraically isomorphic to $\mathfrak{x}_{\mathscr{L}}(M', \mathscr{F}')$, then $\mathfrak{x}(M, \mathscr{F})$ is algebraically isomorphic to $\mathfrak{x}(M', \mathscr{F}')$. Conversely, does $\mathfrak{x}(M, \mathscr{F})$ characterize $\mathfrak{x}_{\mathscr{L}}(M, \mathscr{F})$?

The purpose of this paper is to prove Pursell-Shanks type theorem for certain foliated manifolds.