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On the neighborhood of a compact complex curve with topologically trivial normal bundle

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Introduction

Let C be a non-singular irreducible compact complex curve imbedded in a complex manifold of dimension 2. As an oriented differentiable manifold, the structure of the neighborhood of the curve C is completely characterized by the Chern class of the normal bundle of C, in other words by the self-intersection number (C^2) of C. This topological structure imposes restrictions on the complex analytic properties of the neighborhood of C. Specifically the curve C has a strongly pseudoconvex neighborhood if and only if (C^2) is negative (see Grauert [3]); on the other hand C has a fundamental system of strongly pseudoconcave neighborhoods if (C^2) is positive (see Suzuki [11]).

The purpose of the present paper is to investigate such complex analytic properties of the neighborhood of the curve C when the self-intersection number (C^2) vanishes. We shall see that, if the complex normal bundle N of C is a general element (in the sense of Lebesgue measure) of the Picard variety $\mathfrak{P}(C)$, then C has either a fundamental system of strongly pseudoconcave neighborhoods or that of pseudoflat neighborhoods. We shall find moreover, in the former case, a restriction on the behavior of plurisubharmonic functions and holomorphic functions having singularities along C. This restriction may be regarded as an expression of the weekness of pseudoconcavity of the neighborhood of C.

In §1, we make some preliminary observations concerning flat line bundles, i.e., complex line bundles whose transition functions are constants of modulus 1. In §2, we define the type (1, 2,..., or infinite) for a curve C whose complex normal bundle N is topologically trivial. This type can be described as follows: A unique structure of flat line bundle is introduced on N, and N is extended uniquely to a flat line bundle F over a neighborhood of C; then the type represents the order of coincidence of F and the complex line bundle $[C]$ corresponding to the divisor C. The curve C is of infinite type if F and $[C]$ coincide *formally*. In §3, the case of finite type is treated. We construct a strongly plurisubharmonic function $\Phi(p)$ defined on a neighborhood of C except on C which tends to $+\infty$ as p approaches C. Letting n be the type of C, we can construct, for any real number $n' > n$, such a