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Constructions of eigenfunctions for the Sturm-Liouville operator by comparison method

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§ 1. Introduction

This paper is concerned with constructions of eigenfunctions for the Sturm-Liouville operator $L=-\frac{d^2}{dx^2}+q(x)$ in $(-\infty, \infty)$. Here we assume that the real valued function $q(x)$ satisfies the following conditions:

$$\text{(C)}\quad \begin{cases} q(x) \text{ is piecewise continuous and has the minimum value at } x=x_0, \\ m=q(x_0)=\inf_{-\infty<x<\infty} q(x)<M=\varliminf_{x\to\infty} q(x), \quad (M=\infty \text{ is included.}) \end{cases}$$

Especially we consider concrete constructions of eigenfunctions corresponding to eigenvalues in (m, M), relying upon comparison theorems which assure the existence of bounded solutions $u_+(x, \lambda)$ and $u_-(x, \lambda)$ of $\frac{d^2}{dx^2}u=(q(x)-\lambda)u$ in neighbourhoods of $+\infty$ and $-\infty$ respectively. Namely we try to consider the Sturm's method of comparison even in the case of infinite demain $(-\infty, \infty)$. As we see later, this consideration motivates originally comparison theorems of type stated in Section 2, which are generalized in [2] and [3]. Incidentally we show that there exists a continuous monotone increasing function $\Phi(\lambda)$ satisfying $-\pi<\Phi(m)<0$ such that λ is eigenvalue if and only if $\Phi(\lambda)=(n-1)\pi$, $(n=1, 2, 3,\dots)$. In order to see that appearance of eigenvalues more precisely we need some estimates for $\Phi(\lambda)$. For this purpose we write

$$\Phi(\lambda)=\int_{\Omega(\lambda)}(\lambda-q(x))^{1/2}dx+R(\lambda),$$

where $\Omega(\lambda)=\{x;\ \lambda-q(x)>0\}$ and obtain a suitable estimate for $R(\lambda)$, to show that $R(\lambda)$ is a remainder term as compared with the first term. In many books of physics (for example [1], [5] etc.) we find the following type of formula: $\int_{\Omega(\lambda_n)}(\lambda_n-q(x))^{1/2}dx=\left(n-\frac{1}{2}\right)\pi$, which was explained by the so-called W. K. B. method. As for mathematics Tichmarsh [6] showed that there exists a constant C such that