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On invariant tensors of β-changes of Finsler metrics

By

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Let M^n be an n-dimensional differentiable manifold and $F^n=(M^n, L)$ be a Finsler space equipped with a fundamental function $L(x, y)(y^i=\dot{x}^i)$ on M^n. For a differential one-form $\beta(x, dx)=b_i(x)dx^i$ on M^n, we shall deal with a change of Finsler metric which is defined by

$$(0.1)\qquad L(x, y) \longrightarrow \bar{L}(x, y)=f(L(x, y), \beta(x, y)),$$

where $f(L, \beta)$ is a positively homogeneous function of L and β of degree one. This is called a β-change of the metric. We have specially interesting example of β-change of the metric, for instance,

$$(0.2)\qquad \begin{aligned} &(1)\quad \bar{L}(x, y)=L(x, y)+\beta(x, y),\\ &(2)\quad \bar{L}(x, y)=L^2(x, y)/\beta(x, y),\\ &(3)\quad \bar{L}(x, y)=L^3(x, y)/\beta^2(x, y).\end{aligned}$$

The change (0.2) (1) has been introduced by Matsumoto [12]*. Hashiguchi and Ichijyō [7] named it a *Randers change* and proved a theorem which shows a relation between a Randers change and a projective change.

Next, the change (0.2) (2) is called a *Kropina change*. For a β-change $L\to\bar{L}=f(L, \beta)$, if L is a Riemannian metric $\alpha(x, dx)=(a_{ij}(x)dx^i dx^j)^{1/2}$, then $\bar{L}=f(L, \beta)$ becomes well-known (α, β)-metric ([5], [6]). In particular $\bar{L}=\alpha+\beta$ is a Randers metric ([3], [9]) and $\bar{L}=\alpha^2/\beta$ is a Kropina metric ([11]). Both of them are closely related to physics and so Finsler spaces with these metrics have been studied by many authors, from various standpoint in the physical and mathematical aspect ([3], [9], [22], [23], [26]).

In § 1, we shall study how the fundamental and the torsion tensors change by a β-change of the metric. § 2 is devoted to giving transformation formulas of the torsion and the curvature by a β-change of the metric. In § 3, we consider Randers changes and give some invariant tensors under these changes, and in § 4 we shall study some geometrical properties of these invariant tensors. In §§ 5 and 6, we are concerned with projective Randers changes and also give a characterization of the vanishing Douglas tensor which is invariant under a

* Number in brackets refer to the references at the end of the paper.