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On an isomorphism of the algebra of pseudo-differential operators

By

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I. Let M be a compact connected smooth manifold without boundary, and denote by $L^m(M)$ the space of pseudo-differential operators of order $m \in \boldsymbol{Z}$. We assume that the total symbol of $P \in L^m(M)$ (in each local coordinate) has an asymptotic expansion in homogeneous functions of integer order. Also put $L^\infty(M) = \bigcup_{m \in \boldsymbol{Z}} L^m(M)$, then $L^\infty(M)$ is an algebra over $\boldsymbol{C}$, and $L^{-\infty}(M) = \bigcap_{m \in \boldsymbol{Z}} L^m(M)$ is a two sided ideal consisting of all operators with smooth kernel (for non-compact manifolds see Remark 1 below, and for the definition of pseudo-differential operators see [3] and also for more details see [4] and [5]).

We denote by T^*M the cotangent bundle of M and by T_0^*M the complement of the zero section in T^*M, and also by S^*M the cotangent sphere bundle of M.

Let $\alpha: L^\infty(M) \simeq L^\infty(N)$ be an order-preserving algebra isomorphism, i.e., $\alpha(L^m(M)) = L^m(N)$, for all $m \in \boldsymbol{Z}$, then in [1] Duistermaat — Singer has shown the

Theorem A. *If $H^1(S^*M, \boldsymbol{C}) = 0$, then α is equal to a conjugation by an invertible elliptic Fourier integral operator A: $C^\infty(M) \simeq C^\infty(N)$, that is, $\alpha(P) = A \circ P \circ A^{-1}$ for all $P \in L^\infty(M)$. Here $H^1(\cdot, \boldsymbol{C})$ is the first de Rham cohomology group with coefficients in $\boldsymbol{C}$.*

The canonical relation of this operator A is defined by a homogeneous symplectomorphism C: $T_0^*M \simeq T_0^*N$. If C is defined over all T^*M, then C is the lifting of a diffeomorphism $\mathscr{F}$: $M \simeq N$ (see [6, p. 34]), and the Fourier integral operator A in Theorem A is equal to $\mathscr{F}^*$ up to an invertible elliptic pseudo-differential operator.

On the other hand, in [2] Pursell — Shanks has shown the

Theorem B. *Let i: $X(M) \simeq X(N)$ be an isomorphism between Lie algebras of smooth vector fields on the manifolds M and N. Then the isomorphism i is of the form $i = \mathrm{d}F$, that is, $i(X) = (F^{-1})^* \circ X \circ F^*$, $X \in X(M)$, where F: $M \simeq N$ is a diffeomorphism.*

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