J. Math. Kyoto Univ. (JMKYAZ)
25-2 (1985) 255-267

Analytic families of entire functions of finite order

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

Tetsuo UEDA

(Received December 23, 1983)

Introduction. The classical theorem of Hadamard concerning entire functions of a complex variable is composed of the following three assertions: (i) If f is an entire function of finite order λ, then the order of the zero locus of f does not exceed λ. (ii) For a divisor A of finite order λ prescribed on the complex line $\boldsymbol{C}$, there exists an entire function F of the same order λ having A as its zero locus. F is given by a canonical product of genus q with $\lambda-1\leqq q\leqq\lambda$. (iii) In the same situation as (ii), every entire function f of finite order with zero locus A is written as $f=e^{P}F$ with a polynomial P. The order of f is $\max\{\lambda, \deg P\}$.

Now let Ω be a domain in the space $\boldsymbol{C}^m$ of m complex variables $t=(t^1, \cdots, t^m)$. We consider holomorphic functions f and divisors A on $\boldsymbol{C}\times\Omega$. They can be respectively regarded as families of entire functions and divisors on $\boldsymbol{C}$ depending analytically on the parameter $t\in\Omega$. Their orders are then defined as functions of t. In the present note we will investigate the problem: To what extent do the properties corresponding to the above Hadamard theorem remain valid for these analytic families?

For a function or a divisor on $\boldsymbol{C}\times\Omega$ we consider, along with the order $\lambda(t)$, the regularized order $\lambda^*(t)$ introduced by Lelong [7]. They take on the same value except on a pluripolar set in Ω. We shall find that the concept of regularized order is adequate for our investigation since $\lambda^*(t)$ bounds the rate of growth uniformly in the vicinity of the point t in Ω. Some basic properties of $\lambda^*(t)$ are resumed in § 1.

The central part of our problem concerns with the existence of a holomorphic function of finite order with prescribed divisor A. We want to obtain such a function by forming a canonical product for each $t\in\Omega$. To do this the genus q of the canonical product should be chosen. We wish to choose q independently of the parameter t, while $q+1$ cannot be smaller than the order $\lambda_A(t)$ of the divisor A in order to guarantee the convergence. This is impossible when $\lambda_A(t)$ is unbounded. So we first restrict the variability of t to a subdomain Ω' of Ω on which $\lambda_A(t)$ is bounded, and construct canonical products for $t\in\Omega'$. It is crucial to show that this construction actually yields a holomorphic function