J. Math. Kyoto Univ. (JMKYAZ)
25-3 (1985) 445-457

# Varieties which have two projective space bundle structures

By

Eiichi SATO

(Communicated by Prof. Nagata, Sep. 9, 1983: Revised, May 9, 1984)

## Introduction.

In this article we study the structure of varieties which have two bundle structures whose fibers are projective spaces. Well-known examples are $\boldsymbol{P}^r \times \boldsymbol{P}^s$, $\boldsymbol{P}(T_{\boldsymbol{P}^n})$ and fiber product $R_1 \times_Y R_2$ of $R_1$ and $R_2$ over $Y$ where $R_1$ and $R_2$ are ruled varieties over $Y$. The aim of this work is to classify such varieties under some additional conditions.

Let $M$ and $M_i (i=1, 2)$ be varieties over an algebraically closed field $k$ and let $p$ and $q$ be proper surjective morphisms $M \to M_1$ and $M \to M_2$, respectively, where every closed fiber of $p$ and $q$ is isomorphic to $\boldsymbol{P}^r$ and $\boldsymbol{P}^s$ respectively. To fix the idea let us introduce the following notion:

($P$) We say that $M$ has two projective space bundle structures ($M_1$, $\boldsymbol{P}^r$, $p$; $M_2$, $\boldsymbol{P}^s$, $q$) if there are two varieties $M_1$, $M_2$ and two morphisms $p$, $q$ as above and if $\dim \Phi(M) > \max\{\dim M_1, \dim M_2\}$, where $\Phi$ is the morphism $M \to M_1 \times M_2$ induced by $p$ and $q$ (see Remark 1.6 about the second condition).

Under this notation, we have

**Theorem A.** *Let $M$ be a non-singular projective variety over an algebraically closed field $k$. Assume $M$ has two projective space bundle structures ($\boldsymbol{P}^l$, $\boldsymbol{P}^r$, $p$; $\boldsymbol{P}^m$, $\boldsymbol{P}^s$, $q$).*

1) *If the characteristic of $k$ is zero, then $M$ is isomorphic to either* a) $\boldsymbol{P}^l \times \boldsymbol{P}^m$ *($p$ and $q$ are the first and the second projections, respectively), or,* b) $\boldsymbol{P}(T_{\boldsymbol{P}^l})$, *where $T_{\boldsymbol{P}^l}$ is the tangent bundle of $\boldsymbol{P}^l$.* (*See Lemma* 1.15). *In the case of* (b), $l=m=r+1=s+1$.

2) *If the characteristic of $k$ is positive, additionally, assume that $p$ (or, $q$) is $\boldsymbol{P}^r$-bundle on $\boldsymbol{P}^l$(or, $\boldsymbol{P}^s$-bundle on $\boldsymbol{P}^m$, resp.) in the Zariski topology. Then we have the same conclusion as in* 1).

**Theorem B.** *Let $M$ be a non-singular projective* 3*-fold over an algebraically closed field of characteristic* 0. *Assume that $M$ has two projective space bundle structures ($S_1$, $\boldsymbol{P}^1$, $p$; $S_2$, $\boldsymbol{P}^1$, $q$) with non-singular surfaces $S_1$, $S_2$. Then $M$ is one of the following*

1) $S_1 \times_C S_2$, *where $S_i$ is a $\boldsymbol{P}^1$-bundle over a non-singular complete curve $C$.*