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Excursions in a cone for two-dimensional Brownian motion

By

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1. Introduction and preliminaries.

Let $\{B(t), 0\leqq t<\infty\}$ be the two-dimensional standard Brownian motion process with continuous paths on a probability space $(\Omega, \mathcal{F}, P_x, x\in \boldsymbol{R}^2)$, $P_x(B(0)=x)=1$. We also write P_0 by P. A most significant property of the Brownian paths is known as the *winding property: Let T be a finite Markov time of the two-dimensional Brownian motion process, then with probability one* $\{B(t), T\leqq t<T+\varepsilon\}$ *winds about* $B(T)$ *and cuts itself for every* $\varepsilon>0$ (Ito and McKean [9], 7.11). In this paper we will consider the contrary. Namely, when does occur a *non-winding* in the two-dimensional Brownian paths? We also determine the *law of a non-winding part* by giving the corresponding conditioned limit theorem for the Brownian motion.

Let $\boldsymbol{u}(\theta)$ be a unit vector $(\cos\theta, \sin\theta)$ in $\boldsymbol{R}^2$. For $0<\alpha<2\pi$ we set a circular cone

$$F=F(\alpha)=\{\boldsymbol{x}=r\boldsymbol{u}(\theta): r\geqq 0, 0\leqq\theta\leqq\alpha\}.$$

In the remainder of this section we consider an F such that the vertical angle is $0<\alpha\leqq\pi$ unless anything other is stated. Then F has the following property:

(1.1) *If* $\boldsymbol{x}\in F$, *then* $F+\boldsymbol{x}\subset F$.

Noting the continuity of the Brownian paths, we have from (1.1) the following lemma.

Lemma 1. *Let* $0<\alpha\leqq\pi$. *Then for every* $0\leqq s<\infty$ *there exists the largest interval among all of the closed ones* $[\tau, \upsilon]$ *satisfying*

$$(1.2)\qquad \begin{gathered} 0\leqq\tau\leqq s\leqq\upsilon\leqq\infty \quad \text{and} \quad B(t)\in F+B(\tau) \quad \text{for} \quad \tau\leqq t\leqq\upsilon \\ (\text{or for } \tau\leqq t<\infty \text{ if } \upsilon=\infty). \end{gathered}$$

Let $[\tau(s), \upsilon(s)]$ denote the largest interval, and call it an *F-excursion interval* (*straddling* s) if it does not degenerate to a single point $\{s\}$. We note that the largest interval does not exist in general, when (1.1) does not hold for $\pi<\alpha<2\pi$. Let $\mathcal{E}_F(B)$ be the random set of all F-excursion intervals. Clearly $\{B(t), \tau\leqq t<\upsilon\}$ does not wind about $B(\tau)$, if $[\tau, \upsilon]\in\mathcal{E}_F(B)$. First we consider the following