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The scattering theory for the nonlinear wave equation with small data

Dedicated to Professor Sigeru Mizohata on his sixtieth birthday

By

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1. Introduction and Results

In this paper we consider a small data scattering problem for the nonlinear wave equation

(1.1) $$\partial_t^2 w - \Delta w + f(w) = 0, \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}.$$

Here $2 \leq n \leq 5$, $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ and $f(w)$ represents a nonlinearity which satisfies the following conditions:

(A1) $f \in C^1(\boldsymbol{R})$, $f(0)=0$, $f'(0)=0$;

(A2) $|f'(s_1) - f'(s_2)| \leq C(|s_1|^{\rho-2} + |s_2|^{\rho-2})|s_1 - s_2|$ for $s_1, s_2 \in \boldsymbol{R}$.

In (A2) we have to choose $\rho \geq 2$. Moreover, in the following we require a more stringent condition

(A3) $$\begin{cases} \dfrac{n^2+3n-2+\sqrt{(n^2+3n-2)^2-8(n^2-n)}}{2(n^2-n)} < \rho \leq \dfrac{n+3}{n-1} & \text{for } n=2, 3, 4 \\ \rho = 2 & \text{for } n=5. \end{cases}$$

Scattering theory compares the asymptotic behaviors for $t \to \pm\infty$ of solutions of (1.1) with those of the free wave equation

(1.2) $$\partial_t^2 w - \Delta w = 0, \quad (x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}.$$

The comparison will be done in the energy space. For $s \in \boldsymbol{R}$ and $1 \leq p \leq \infty$, let $H^{s,p} = H^{s,p}(\boldsymbol{R}^n)$ and $\dot{H}^{s,p} = \dot{H}^{s,p}(\boldsymbol{R}^n)$ be the Sobolev spaces which are the completion of $C_0^\infty(\boldsymbol{R}^n)$ with norms

$$\|u\|_{H^{s,p}} = \|\mathscr{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)]\|_{L^p}$$

and