J. Math. Kyoto Univ. (JMKYAZ)
25-4 (1985) 687-691

Remarks on generalized rings of quotients, IV

By

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(Received, Aug. 22, 1984)

It is known that if R' is a generalized ring of quotients of a ring R contained in the total quotient ring $T(R)$ of R, then for any ring R'' with $R \subseteqq R'' \subseteqq R'$, R' is a generalized ring of quotients of R''. In other words, if R' is R-flat, then R' is R''-flat, for R', R'' such that $R \subseteqq R'' \subseteqq R' \subseteqq T(R)$. (Corollary 1 to Theorem 1 in [1] and Lemma 2 in [11]).

In this paper, we shall show that the converse, in a sence, to the above is valid if R' is a domain. Precisely speaking, let $R \subseteqq R'$ be rings such that R' is R-flat. Consider the following condition (F):

(F) For any ring R'' with $R \subseteqq R'' \subseteqq R'$, R' is R''-flat.

It is clear that if R' is a field, then R' satisfies (F) for any subring R of R'. We shall show the following:

Assume that R' is a domain. If R' satisfies (F) and if $R' \subsetneqq T(R)$, then R' is a field. (Theorem 2.7 in §2)

In this paper, first we shall give some results on flatness of rings in §1. In §2, we shall prove the main result of this paper and in §3 we shall give some results in the general case. Notation is the same as in [1], [2] and [3]. A pair (R, R') means that R' is a ring and R is a subring of R'.

§1. We shall begin with some results on flatness and on the condition (F).

Lemma 1.1. *Let (R, R') be a pair and let $\mathfrak{a}'$ be an ideal of R' containing a non-zero-divisor. Let $R''=R+\mathfrak{a}'$, which is a subring of R' containing R such that $T(R'')=T(R')$. Assume that R' is R''-flat. Then we have $R'=R''$ if R' is integral over R or if $R=k$ is a field.*

Proof. If R' is integral over R, then R' is also integral over R'' and we have $R'=R''$ by Corollary 2 to Theorem 1 in [1]. Assume that $R=k$ is a field. Suppose that $R' \neq R''$. Then there is an $r' \in R'$ such that $r' \notin R''$. Since $r' \notin \mathfrak{a}'$, it is easily seen that $(R'': r')=\mathfrak{a}'$ and $\mathfrak{a}'R'=\mathfrak{a}' \neq R'$ which contradicts Theorem 1 in [1]. Thus we have $R'=R''$.

Lemma 1.2. *Assume that a pair (R, R') satisfies (F). Then*: (1) For any ring R_1 *such that $R \subseteqq R_1 \subseteqq R'$, the pair (R_1, R') also satisfies (F).*