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Spectral asymptotics for elliptic second order differential operators

By

Georgi POPOV

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0. Introduction and statement of the results

Let X be an open set in $\boldsymbol{R}^n$, $n \geq 2$, with a smooth boundary Y and $\boldsymbol{R}^n \backslash X \subset B_R = \{x; |x| \leq R\}$ for some $R > 0$. Suppose that $\boldsymbol{R}^n$ is provided with a smooth Riemannian metric $ds^2 = g^{ij}(x)dx_i dx_j$ which is Euclidean outside the ball B_R. Set $g(x) = \det(g_{ij}(x))$, $g_{ij}g^{jk} = \delta_i^k$ (Kronecker's index), $i, k = 1, \dots, n$ where the summation convention is used. Let Δ_g,

$$\Delta_g u(x) = g(x)^{-1/2} \frac{\partial}{\partial x_i} \left(g(x)^{1/2} g^{ij}(x) \frac{\partial u}{\partial x_j}(x) \right), \; u \in C^\infty(\boldsymbol{R}^n),$$

be the corresponding Laplace-Beltrami operator and $H = -\Delta_g + V(x)$ with some real-valued function $V \in C_0^\infty(B_R)$.

The operator H will be considered as a self-adjoint operator in $L^2(X)$ (the scalar product in $L^2(X)$ is given by $(u, v) = \int_X u(x)\overline{v(x)} g^{1/2} dx$) with boundary conditions of Dirichlet (Neumann) type on Y. Throughout this paper $B_D u = u_{/Y}$ while Neumann boundary condition is of the form $B_N u = \left(\frac{\partial u}{\partial \nu}(x) + \gamma(x) u(x) \right)_{/Y} = 0$, where $\gamma \in C^\infty(Y)$ is a real-valued function and ν is the outward normal to Y, pointing into $\boldsymbol{R}^n \backslash X$.

The spectral function $e(\lambda; x, y)$, $\lambda \in \boldsymbol{R}^1$, of the operator H is determined as the distribution kernel of the spectral projectors E_λ of H. Namely,

$$(E_\lambda u, v) = \int_{X \times X} e(\lambda; x, y) u(y) \overline{v(x)} g(x)^{1/2} g(y)^{1/2} dx dy, \quad u, v \in C_0^\infty(X).$$

This function is closely related to the outgoing (incoming) Green's functions $G^+(\lambda; x, y)(G^-(\lambda; x, y))$ which are given by

$$(0.1) \qquad (-\Delta_g + V(x) - \lambda^2) G^\pm(\lambda; x, y) = \delta_y(x) \quad \text{in} \quad X, \quad y \in X,$$

$$(0.2) \qquad B_j G^\pm = 0 \quad \text{in} \quad y \in X \quad \text{and} \quad j = D \quad \text{or} \quad j = N,$$

and by the outgoing (incoming) Sommerfeld's condition at infinity