J. Math. Kyoto Univ. (JMKYAZ)
26-1 (1986) 39–64

Half-canonical divisors on variable Riemann surfaces

To Professor Yukio Kusunoki on his sixtieth birthday

By

Clifford J. EARLE and Irwin KRA[1)]

1. Introduction and first statement of main result.

1.1. The study of holomorphic sections of the Teichmüller curve was initiated by Hubbard [16] (see also Earle-Kra [7], [8]). Hubbard proved that for $p=2$ the map $\pi: V_p \to T_p$ has precisely 6 holomorphic sections, the Weierstrass sections, while for $p \geqq 3$, π has no holomorphic sections. The base space T_p is a contractible domain of holomorphy. Nevertheless, Hubbard's result shows that for $p>2$ we cannot choose a point on each surface in a way that depends holomorphically on moduli. On the other hand, we can choose on every surface of genus $p \geqq 2$ a divisor class of degree one that depends holomorphically on moduli (see [5]).

1.2. Let n be a positive integer. Let $\pi_n: S_T^n(V_p) \to T_p$ be the fiber space whose fiber over $t \in T_p$ is $\pi_n^{-1}(t) = S^n(X_t)$, the n-fold symmetric product of the Riemann surface $X_t = \pi^{-1}(t)$ represented by t (see §§2 and 3 for details). The points of $S^n(X_t)$ can be identified with the integral divisors of degree n on X_t. A holomorphic section of π_n corresponds to a choice on each surface of an integral divisor of degree n that depends holomorphically on moduli.

In this paper we concentrate on the case $n=p-1$. For $n<p$ every divisor $D \in S^n(X)$ on a compact Riemann surface X of genus p is *special* in the sense that there exists on X a nontrivial abelian differential of the first kind that vanishes on D ($p-1$ is the largest integer with this property). A divisor $D \in S^{p-1}(X)$ is *half-canonical* if $2D$ is the divisor of a nontrivial abelian differential of the first kind. Similarly, a section s of π_{p-1} is *half-canonical* if $s(t)$ is a half-canonical divisor for all $t \in T_p$. We can now state our main result as

Theorem 1. *The map $\pi_{p-1}: S_T^{p-1}(V_p) \to T_p$, $p \geqq 2$, has a half-canonical holomorphic section if and only if $p=2$, 3, or 4. The number of such sections is precisely 6 for $p=2$, 28 for $p=3$, and 120 for $p=4$; that is, precisely the number of odd half-periods in the Jacobi variety.*

Communicated by Prof. Kusunoki Oct. 25, 1984.

1) This research was partly supported by National Science Foundation grants MCS–8301564 (first author) and DMS 8401280 (second author).