J. Math. Kyoto Univ. (JMKYAZ)
26-1 (1986) 13-22

Holomorphic families of Riemann mapping functions

To Yukio Kusunoki on his 60th birthday

By

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1. Introduction.

Suppose a closed Jordan curve γ_0 in the plane is deformed into other closed Jordan curves γ_λ, where λ is a complex variable in the unit disk $\boldsymbol{D}$. Furthermore, suppose the deformations depend holomorphically on λ. In such a situation one is naturally led to investigate the dependence on λ of suitably normalized Riemann mapping functions f_λ of the unit disk onto G_λ, the inside region of γ_λ. For a summary of earlier results on this topic see Warschawski [8]. A result of Rodin [5] shows that, in general, f_λ depends real analytically on the real and imaginary parts of λ. It is of interest to know when this real analytic dependence is actually complex analytic (i.e., holomorphic in λ). In the present paper we give some methods and results which have been useful in our preliminary investigation of the question.

2. Definitions.

Throughout this paper $\boldsymbol{D}_r$, where $r>0$, denotes the disk $\{z\in \boldsymbol{C}: |z|<r\}$. The Riemann sphere will be denoted $\hat{\boldsymbol{C}}$. Let $E\subset\hat{\boldsymbol{C}}$. A map

$$F: \boldsymbol{D}_r\times E \longrightarrow \hat{\boldsymbol{C}} \tag{2.1}$$

is called a *holomorphic motion of E* if the following three conditions are satisfied:

(i) For all $\lambda\in\boldsymbol{D}_r$ the map $F(\lambda,\cdot)$: $E\to\hat{\boldsymbol{C}}$ is injective.

(ii) For all $z\in E$ the map $F(\cdot, z)$: $\boldsymbol{D}_r\to\hat{\boldsymbol{C}}$ is holomorphic.

(iii) For all $z\in E$, $F(0, z)=z$.

The "λ-lemma" of Mañé-Sad-Sullivan [4] states that the holomorphic motion (2.1) extends to a holomorphic motion

$$\hat{F}: \boldsymbol{D}_r\times Cl\,E \longrightarrow \hat{\boldsymbol{C}} \tag{2.2}$$

of the closure of E. Furthermore, for each $\lambda\in\boldsymbol{D}_r$ the map

$$\hat{F}(\lambda,\cdot): Cl\,E \longrightarrow \hat{F}(\lambda, Cl\,E) \tag{2.3}$$

is a homeomorphism and is quasiconformal on every open subset of $Cl\,E$.

Communicated by Prof. Kusunoki Oct. 23, 1984
* Research partially supported by the National Science Foundation.